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ABSTRACT

This report is intended to provide attention to issues that the Cambridge Conference feels are related to the mathematical training of elementary school teachers. The document is divided into three parts. Part I, titled "The Problems and Proposals Towards the Solution," contains the following chapters: (1) Introduction to the Report; (2) The Preservice Training of Elementary School Teachers; (3) The In-Service Environment of Elementary School Teachers; and (4) Recommendations. Part II, "Details of the Mathematics Curriculum," contains: (5) Details of Proposal One; and (6) Details of Proposal Two. The third part is titled "Essays on Some Aspects of Elementary School Training," and contains six essays. It is hoped that this document will inspire debate, controversy, and experiment, and that out of these will eventually emerge guidelines which can actually be used. (MP)

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Goals for
MATHEMATICAL EDUCATION OF ELEMENTARY SCHOOL TEACHERS

A Report of The Cambridge Conference on Teacher Training

PUBLISHED FOR EDUCATION DEVELOPMENT CENTER, INC.
Newton, Massachusetts

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CAVEAT

This report, like the report of the 1963 Cambridge Conference on School Mathematics, *Goals for School Mathematics*, should be regarded only as a tentative and, hopefully, thought-provoking document. It is addressed, in the main, to those directly concerned with the mathematical training of elementary school teachers: professors in universities and colleges, chairmen of mathematics departments, and deans of schools of education. It might also be of interest to many school teachers and school administrators, but, it most definitely should not be regarded as a prescription for them to act upon. Since future mathematical curricula are, themselves, so far from determined, it would be preposterous to attempt to define, in any firm terms, the training of those expected to teach these curricula. What is hoped is that the report will inspire debate, controversy, and experiment, and that out of these will eventually emerge guidelines which can actually be used.

In preparing the report, an attempt has been made to avoid use of the words "should" and "recommend," particularly as they might apply to specific programs of study. If the reader encounters them or similar words, in print or by implication, he is urged to consider them in the light of what has been said above; he has run into something which the Conference calls to his attention and feels he should consider seriously.

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The Cambridge Conference on Teacher Training, which took place from June 13 through July 8, 1966, was organized and administered by Educational Services Incorporated, 55 Chapel Street, Newton, Massachusetts, under a grant from the National Science Foundation. On January 20, 1967, Educational Services Incorporated merged with the Institute for Educational Innovation to form a new organization, Education Development Center, Inc.

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PART ONE

The Problems and Proposals towards the Solution

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Chapter I Introduction to the Report

SECTION 1

Preliminary Remarks

The wave of reform that has swept across the teaching of mathematics and the physical sciences in American schools has stimulated a public response which, though not always comprehending, has been in the main a helpful and healthy one. Parents of school children learning the new ways are somewhat bemused, perhaps, but have displayed a willingness to trust the experts, at least up to some thus far undefined point. The press has made its little jokes about Johnny's queer answers for two-times-two, and has been humorously condescending about Dad's inability to help with the homework. Still, the American people, if one can judge from the number of tax dollars spent on curriculum reform in the past eight years, have wanted, or have been willing to see, fundamental changes made in the subject matter being taught to their children. Responding perhaps to an unvoiced sense of having missed something important in their own school days, parents, in a manner quite uncharacteristic of the American generations preceding them, have stood by approvingly while university professors have intellectualized the content of elementary and secondary school mathematics courses. They have gone even further on their own, buying by the thousands hastily turned out books — a few good, some passable, most of them bad — that purport to bring Dad and Mom up to the mark in the "new" math and so enable them to trail along with Junior in his progress through the world of the mind.

But in all this public interest in the reform of school curricula — an interest concretely evidenced in shelf upon shelf of books and column upon column of newspaper and magazine type — there has been a conspicuous lack of attention given to the ultimately crucial problem, which this present report will consider only in relation to the teaching of mathematics. All the thousands of teachers who must be capable of dealing with new subject matter: where are they to come from?

If any one conclusion can be drawn from the experience of the past few years, it is that for the foreseeable future the teaching of mathematics in American schools has broken its traditional bonds. Not in our lifetime, or in the lifetime of today's kindergarten pupils, and probably not for centuries to come, if ever at all, will school mathematics return to the rigid mold it occupied for nearly two hundred years. It will change and spread and grow as mathematics always has done, and, more likely than not, the rate of change will increase rapidly under the impact of the computer. Where does the teacher stand in this flux? How do we teach him or her today to pass on tomorrow's mathematics to the pupils? How do we teach tomorrow's teacher to teach the mathematics of the day after tomorrow?

Mathematicians have no simple answer to these questions, but they have been aware of and worried by the implications. The retraining of teachers has been a major concern from the earliest days of the University of Illinois Committee on School Mathematics (UICSM) and the School Mathematics Study Group (SMSG). The National Science Foundation (NSF) has provided money for dozens of summer and in-service institutes where the nation's teachers have been able to acquire a more or less easy familiarity with the new methods and materials suddenly thrust upon them. The Teacher Training Panel of the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America made formal recommendations for the training of mathematics teachers at all levels and energetically sought to get these recommendations implemented through reports, writings, and many conferences. Among the most significant CUPM activities has been the sponsorship of a series of conferences on elementary teacher training in all the states. Indeed, a historian may one day record that the principal effect of the first wave of curriculum reform in the United States was not the improvement (if improvement existed) in the pupils' performance but rather the broadening of the teachers' experience and the deepening of their intellectual resources. The public, understandably concerned only with the immediate success or failure of its children, has ignored this aspect of the changes in our schools. Except in our most sophisticated cities and suburban areas, we Americans are inclined to take for granted the teachers' ability to teach, and, personalities and politics apart, would no more question the competence of a teacher in the classroom than we would doubt the competence of a pilot at the controls of his transoceanic jet liner.

Until now the solutions of the problems of retraining teachers to fit them to new methods and new materials have been *ad hoc* solutions, intended only to tide us over the period of transition, and it does not take an unduly suspicious nature to detect an air of complacency about the schools of education. There is, to be sure, cause for a certain amount of gratification that the nation's teachers of mathematics should have done as well as they have in wrenching themselves out of old ways of thinking, but their accomplishments, real though they are, are soon going to be proved insufficient. Psychological factors operate here, as well as pedagogical and mathematical difficulties.

One of the problems of educational reform is that any attempt to reduce routine, rigidity, and ritualization in instruction may well be frustrated by the crucial influence of teachers who may transmit those very qualities from their own school experience. Even if higher education is capable of altering the cycle, it ordinarily does not, and the most effective incentive for change in collegiate programs for the instruction of teachers and prospective teachers is a change in the schools' own demands on their teachers in service. Thus improvement in the qualifications of teachers is essential for school reform, and reform, conversely, is essential to improvement in teachers. The important lesson is that steady curriculum and pedagogical development in the schools is vital to improvement in the selection and preparation of teachers.

It would be soothing to suggest that this progression of demand and constructive response can continue indefinitely; but in a subject as complex as mathematics there exists a limit on how much even the most dedicated and talented teacher, short of time and harried by administrative obligations, can do in his off hours to repair the shortcomings of his own professional preparation and to inch himself forward into untried intellectual fields. Except for the extraordinarily gifted or the unusually well prepared, a time arrives when the study of mathematics becomes difficult. In the past this point has usually been well beyond the compass of the American elementary or secondary school teacher; but who is to define today the extent of his domain? All the signs, abroad as well as in the United States, seem to herald an impending rush forward. The UICSM and SMSG courses, which the visibly awed public has lumped together as "new math," may in a very few years seem tame indeed when compared with the mathematical fare of quite young children. The community of mathematicians is deeply concerned; and this report represents, in an admittedly tentative manner, the exploratory efforts of one group to anticipate an apparently inevitable trend.

SECTION 2

Background of The Cambridge Conference on School Mathematics

In the summer of 1963, a group of mathematicians gathered in Cambridge, Massachusetts, at the invitation of Educational Services Incorporated, and with support from the National Science Foundation, to discuss the current state and the possible future of mathematics in the elementary and secondary schools. It was not their purpose to evaluate the progress made under the new curricula developed by UICSM, SMSG, and others; they were looking ahead to what might be accomplished in a second wave of reform after the new courses and fresh attitudes toward mathematics had been established solidly in the schools and accepted by the public at large.

The group, which met for several weeks, adopted the name Cambridge Conference on School Mathematics (CCSM) and issued a formal report, *Goals for School Mathematics,* which was published in 1963 by Houghton Mifflin Company for general distribution. That small book (which will be referred to hereafter in this report as *Goals*) attracted an unexpected amount of attention among mathematicians and educators, both in the United States and abroad, and provoked lively discussion, pro and con, that has continued over the intervening three years.

The debates inspired by *Goals* centered on the set of proposals it put forth for the mathematics curricula of two or three decades hence, from kindergarten through Grade 12. Those proposals were perhaps too general to be called recommendations; rather, they represented the thinking of only one particular group of mathematicians in an optimistic frame of mind. The subject matter they outlined for the pupils of the 1980's can be described roughly as calculated to bring high school graduates to a mathematical level now attained only at the end of the third year of college. On first reading, three years ago, the *Goals* proposals sounded radical,* to say the least, but the passage of time, along with growing public awareness of the necessity of accelerating mathematical education to keep up with technological and social change, has noticeably softened the shock. Experimental work of an even more drastic nature than the *Goals* proposals already has been started in the United States, and the mathematicians of at least one European country, Denmark, have embarked on formal development of a school mathematics program that seems to parallel closely much of the *Goals* outline.

Since it was the intention of the members of the 1963 Conference only to explore and to stimulate, they put off to another day, or for another group to take up, all the specific considerations that would require thorough examination before the *Goals* proposals could be implemented. The report listed suitable topics for two versions of K-12 curricula and appended correlative supporting material, but did not attempt to suggest how one might go about incorporating the proposals into the present American school system or how one would train teachers to teach the more difficult topics.

For a curriculum that would accelerate by three years the progress of pupils through K-12 mathematics, the problem of training teachers becomes critical. The mathematical content proposed for the high school years would be beyond the present capability of today's teachers. The curriculum outlined for the elementary grades would not appear, on the surface, to be comparably taxing mathematically, but it would demand a classroom attitude that most teachers could be expected to acquire only through a system of training considerably different from that which is now standard in the United States. The fundamental view of the 1963 Conference was that the traditional drill-for-drill's-sake should be abandoned and that manipulative skills should be taught instead by the adroit introduction of problems whose primary purpose would be to illustrate mathematical concepts. To expect the practicing or prospective teacher to take this point of view on his or her own, without benefit of thorough instruction along new lines of thinking, would be to ask too much of human nature. While the outstanding work of the Committee on the Undergraduate Program in Mathematics has given promise of alleviating the shortage of teachers adequately trained to handle UICSM and SMSG materials, and other material recently developed, further steps must be taken in the near future if the *Goals* projection of the mathematics curriculum of a couple of decades hence is the accurate one, or even approximately the right one.

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* But not to Professor Marshall H. Stone. See *The Mathematics Teacher,* April, 1965, pp. 353-360.

After issuing their report the members of the 1963 Conference disbanded, but its Steering Committee met in 1964 and decided to continue as a more or less permanent organization. The following summer (1965) it brought mathematicians to Cambridge to work for several weeks on development of experimental units pertinent to the *Goals* proposals, and it further decided to organize another conference that would get down to the concrete task of tackling the problem of training teachers to deal with *Goals* curricula. This meeting was set for the next summer (1966) under the aegis of the Education Development Center, Inc. It was from this background that the Cambridge Conference on Teacher Training (CCTT), with whose activities this report is concerned, came into being.

SECTION 3

Summary of the Report of The Cambridge Conference on School Mathematics

The twenty-five mathematicians who attended the 1963 Conference in Cambridge represented both universities and industry. Their fields of interest included algebra, geometry, topology, analysis, statistics, applied mathematics, physics, and chemistry.

It was agreed from the outset that, in setting goals for mathematics curricula, the 1963 Conference would have to defer consideration of the serious and closely related problem of teacher training until its first task was completed. The Conference also took account of the possibility that there might be intrinsic limitations on the ability of young children to handle mathematical ideas; it felt, however, that the boundaries of those limitations, if they exist, are not well defined, and that there is as yet little evidence concerning the degree to which they can be changed by the teaching process. Recognizing then that its work was necessarily of a tentative nature, the Conference turned to its main objective, the curriculum from K through 12.

The 1963 Conference found itself essentially in complete agreement on the mathematical aims of the elementary school.

Through the introduction of the number line, the child would be started immediately on the whole real number system, including negatives. To be sure, he would have formal names at first only for integers and the simplest rational numbers, but all his work would keep him aware of the existence of other numbers, and of the fact that they too have sums, products, etc. By this wedding of arithmetic and geometry at the pre-mathematical level, the intuition of the child would be developed and exploited, and the significance of the arithmetical operations clarified. A child provided with these complementary viewpoints would have a very good chance to understand the essential nature of mathematics and its relationship to the "real" world.

The order properties of the real number system would be studied from the beginning, and would be used in inequalities, approximations, and order of magnitude estimates.

The use of Cartesian coordinates ("crossed" number lines) would begin almost as soon as study of the number line itself. Moreover, in agreement with Freudenthal and other pioneers, an early development of the child's spatial intuition is considered essential. Study of the standard shapes in two and three dimensions would continue concurrently, and would include discussion of their symmetries.

The notions of function and set should be used throughout; of course, set theory and formal logic should not be emphasized as such, but the child should be able to build his early mathematical experience into his habitual language. Informal algebra should be taken up along with the arithmetic operations.

The 1963 Conference agreed that reasonable proficiency in arithmetic computation and algebraic manipulation is essential to the student of mathematics. But this is not an argument in favor of a curriculum devoted primarily to computation with contrived numbers through the whole of grammar school. Long pages of addition and multiplication problems add nothing to a student's understanding of the processes involved; nor do they teach him *when* to add or multiply. At best, they improve the computational speed of a student who understands how to do the algorithms (an objective that by itself had little appeal to the members of the Conference); at worst, they dissipate or destroy the interest that a good student has in the subject. Entirely adequate practice in computation can be built into problems that, on their own merits, genuinely attract the student's interest.

Because of both the intuitive appeal and the basic importance there should be an introduction to the elementary ideas of probability and statistical judgment, accompanied by concrete experimentation with random processes.

The concern for motivation, applications, and the interplay between mathematics and the physical world, was a constant theme of the 1963 Conference report, subject, of course, to the constraints of the pupils' limited experience of science in the elementary school. Geometry, however, offers a rich area within which students can explore the relation between physical objects and their idealized mathematical abstractions. As the students' experience deepens, it should be possible to introduce more sophisticated models.

Having studied arithmetic and geometry, mostly informally, in the elementary school, the student, according to the *Goals* vision of his progress, will be prepared for a sound treatment of geometry and the algebra of polynomials, beginning in Grade 7. The mathematics curriculum for the secondary school can therefore go much further than it commonly does at present. The program of a student electing mathematics each year will, at the end of the twelfth year, have contained a closely-knit presentation of calculus, linear algebra, and probability, involving a brief introduction to other mathematical topics.

The 1963 Conference did not reach any substantial agreement as to the order of presentation or the specific content for this program. Indeed, the multitude of sound proposals suggest that there is no unique optimal solution. Two arrangements of the material proposed for the secondary level were developed in some detail.

The 1963 Conference also arrived at other recommendations which dealt more with methods of presentation than with specific mathematical content. It was felt, for example, that it was desirable to adopt the "spiral" approach, in which every new topic is introduced early under low pressure and is then reconsidered repeatedly, each time with more sophistication, and each time showing more of its interconnections with the rest of the subject. The result should be a sort of guided tour of mathematics. This approach has many important advantages. In the first place, the basic unity of the subject is stressed automatically. Moreover, in the upper grades this approach assures that the student will be exposed concurrently to a mixture of intuitive "pre-mathematics" and rigorous mathematics. Provided that the distinction is made clear to the student, this mixture will give a much more honest picture of what mathematics is, an organism continuously growing through the interaction of intuition and logical analysis, rather than a static structure walled in sterile rigor.

The same precept led, secondly, to the suggestion that topics should receive multiple motivation. During the pre-mathematical stage of some topics, it may be wise to give several different informal presentations, each leading up to the desired goal (for example, the rules for multiplication of negative numbers), rather than to leave students with the feeling that there is only one correct road. Ideally, such active exploration should help to convey to the student the important fact that mathematics is something one *does*, not something that one absorbs passively. One would hope to strengthen the impression that a mathematical idea appeared first as someone's solution of a problem. The problems thus become a matter of importance equal to, or even greater than, that of the textual material itself. The idea therefore was advanced that the design of imaginative problem sequences involving combinations of routine techniques and "discovery" procedures was a matter of the greatest importance in curricular development.

There is much that must be done before the ideas in that proposal can be implemented. Some suggestions in the *Goals* report are being tried already in some of the current educational experiments, either piecemeal or as part of a more extensive program. There must be many further experiments to determine just what is possible at any given age level. Texts and supplementary materials will have to be written. Unquestionably the most difficult problem lies in the training of teachers, upon whom the success or failure of curricular reform ultimately rests. Nor is this an isolated problem, for the pressure to advance our mathematical goals is being felt at all levels of the profession, and a single part of the problem cannot be solved in isolation from the others. If the proposals formulated by the 1963 Conference are to become a reality within the foreseeable future, it is necessary that the entire mathematical community devote considerable attention to the training of teachers at all levels.

SECTION 4

The Cambridge Conference on Teacher Training

Like the 1963 Conference, the Cambridge Conference on Teacher Training (CCTT), which met in 1966 at the Pine Manor Junior College in Brookline, Massachusetts, was composed principally of persons holding university positions or the equivalent in pure or applied mathematics or statistics. Several school teachers from the University of Illinois Arithmetic Project, currently working with the Education Development Center at Newton, Massachusetts, attended the sessions. The conference membership included three British mathematicians, two of whom had been associated with the School Mathematics Project (at the University of Southampton), which was the first of several groups that have undertaken to produce materials for reform of the mathematics curricula in British schools. For most of the Americans the presence of these three at the meeting and the display of SMP* books and books from other British projects, including the Midlands Mathematical Experiment and the St. Dunstan's College Series,** constituted an introduction to what has been done in mathematics reform in the British Isles. While it would be difficult or impossible to pinpoint specific examples, it would not be too much to say that the quite different approach of the British to teaching mathematics, and the wholly different character of their textbooks, exerted a considerable influence on some of the Americans at the meeting. The 1966 Conference membership included also eleven members of the 1963 Conference, but only two were able to spend substantial time at the 1966 meeting.

The majority of the participants spent four weeks at Pine Manor Junior College in an atmosphere of hard work and strenuous discussion, if not of mutual agreement. To imply that they came together in one mind and remained so would be to distort the facts. Elaborate philosophical and pedagogical structures painstakingly erected over periods of several days were demolished at single plenary sessions. Divergence of opinions broke apart organizational groups and led to formation of new groups. On all sides misgivings were expressed as to the propriety and the value of any suggestions university mathematicians might offer about the teaching of mathematics in an environment as remote from their own professional experience as the elementary school classroom necessarily must be. But in the end the membership polarized around two fairly distinct points of view and produced (within the self-imposed restrictions of the *Caveat* at the beginning of this report) the outlines of two curricula which they believed would adequately prepare teachers to cope with any elementary school mathematical materials that may be developed to fit the *Goals* proposals. It is interesting and perhaps significant that, although the two outlines were based on quite different premises, they turned out to have many common points and, as will be seen in the detailed presentation in Part Two of this report, they differ less than might be expected of two programs developed independently.

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* The School Mathematics Project will be referred to as SMP. Its books are published by the Cambridge University Press.

** In the United States, the St. Dunstan's College Series appears in paperback under the imprint of Houghton Mifflin Company (1965). Attention is directed particularly to *Matrices*, Volumes 1 and 2, by Geoffrey Matthews, who now heads the Nuffield Mathematics Teaching Project, one of the major British contributions to curriculum reform.

The working papers of the various organizational groups of the 1966 Conference, and of the individual members, piled up into a sizeable bulk of manuscript, which, in the two weeks following the formal sessions, was winnowed by a small writing group under the supervision of the chairman, John Green. The results of their editorial efforts composed the materials on which this report is based. The following paragraphs are intended to give some orientation towards the content.

A principal reason for placing emphasis on the preservice training of elementary school teachers is that the young female teacher has a very short vocational-life expectancy. Within three or four years after receiving the bachelor's degree, a large percentage are not teaching. Some return to the classroom two to twenty years later, when family demands have diminished. Their teaching certificates in most cases have remained valid. Thus for many teachers, the quality of their preservice training will have a powerful and often decisive influence on the effectiveness of their teaching, not easily changed or supplemented in the short time available to them for further training after the bachelor's degree.

The two curricula proposed in this report will make heavy demands on the students. Members of the 1966 Conference had strong feelings on this point. Elementary teachers of the future will need courses of the kinds to be described, and from the large number of well-motivated students who will enroll in elementary teaching programs a substantial percentage will be able to absorb from those courses not only the mathematical skills they require but some of the attitudes, interest, and enthusiasm for mathematics which are essential for effective teaching.

Chapter III of Part One is concerned with the in-service environment of elementary school teachers. For elementary school teachers mathematics is only a part of the total intellectual demands to be met in their work. Without unduly increasing the mathematical responsibilities of the teachers, there are ways to improve the quality of teaching in the schools by use of elementary mathematics specialists and technological aids. The problems of enlisting the help of professional mathematicians in both teacher education and in helping teachers with their work in the elementary schools are also discussed. The report does not give adequate consideration to the work in educational psychology and testing needed to guide and evaluate the evolving patterns of learning and teaching, both for elementary school students and their teachers. The Conference wishes to draw attention to this important area for further study.

Chapter IV of Part One contains specific recommendations for implementing the proposals. These involve the preparation of text materials, communication of the recommendations to school administrators and to the faculties of teachers colleges and universities, and the problems of continuing the education of elementary school teachers.

Part Three of the report consists of a series of essays on various problems of elementary teaching, written by various members of the Conference.

Chapter II The Preservice Training of Elementary School Teachers

SECTION 1

Assumptions of the Proposals

Both proposals for the general mathematical training of elementary school teachers start from the assumption that students entering the program will have completed Grade 10 SMSG mathematics, or the approximate equivalent. It is further assumed that students could enroll in the program for mathematics specialists either after completing the general program for elementary school teachers, or with a background of the usual first two years of a liberal arts mathematics major.

Relation to CUPM Recommendations

The two curriculum proposals for the generalist elementary school teacher are laid out in two-year sequences, or twelve semester hours. Whether the mathematics contained in them can be taught adequately in twelve semester hours is admittedly not known; only long and much more careful consideration, accompanied by experimentation, can tell us that. It may turn out that the price for obtaining teachers capable of handling the *Goals* material will be fifteen or more semester hours of preparation.

The CUPM recommendations (Level 1) for elementary school teachers also provide for twelve semester hours of mathematics in college and are described concisely in the following quotation from their 1961 report entitled *Recommendations for the Training of Teachers of Mathematics.*

> As a prerequisite for the college training of elementary school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assured that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of the following courses:

(A) A two-course sequence devoted to the structure of the real number system and its subsystems.
(B) A course devoted to the basic concepts of algebra.
(C) A course in informal geometry.

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view.

The 1966 Conference does not consider its proposals to be in conflict with the commendable CUPM recommendations, which it regards as positive and immediately applicable. There is considerable overlap between the material in the CUPM recommendations and the CCTT proposals, although the latter go considerably beyond and involve more mathematical sophistication than the former. The CCTT proposals are aimed (hopefully) at a time when the CUPM recommendations will have been strongly implemented, and the new generation of potential teachers, coming into the college from improved courses in the high schools, will have profited by that implementation. Of interest is the following quotation from CUPM Report Number 9: *Ten Conferences on the Training of Teachers of Elementary School Mathematics.*

> It is clear that if this (*Goals*) is to be the curriculum of the future, our Level 1 recommendations are only makeshift, and represent only the best we can do with the students now entering the schools and with the present staff engaged in teacher training. In the future, we will have to do a good deal better.

It is also of interest to note that four members of the CUPM Teacher Training Panel participated in the 1966 Conference.

SECTION 2

Proposal One for the Training of Generalists

The group at work on this proposal decided, after some debate, that it would be sensible to elaborate the *Goals* outline of a K-6 curriculum before attempting to lay down a program for the training of teachers who would be expected eventually to teach the *Goals'* topics. The exercise of systematically filling out the *Goals'* skeleton, they believed, would give a better perspective and might afford insight into possible new approaches that otherwise would not be suggested. The group's elaboration of the *Goals* proposals for K-6 appears as Essay 1 in Part Three.

This tying of the proposal to the K-6 curriculum does not imply that the training program should be merely an upgraded K-6 sequence, or that topics should be ignored or excluded if they do not contribute directly, in terms of mathematical content, to future classroom activities. It does imply that a college professor, in preparing his materials for teachers, should know in detail what they are going to teach and keep constantly in mind the interrelationships involved.

With this principle and this particular curriculum in mind, the group prepared a list of topics and ideas which the members believed an elementary school teacher should be thoroughly familiar with, and then outlined a small number of mathematical packages which they hoped would capture the imagination of the prospective teacher, deepen insight, and, at the same time, include all the necessary topics. Courses A (one year), B (one-half year) and C (one-half year) were the result. Those three courses are offered by the 1966 Conference as one example of a two-year sequence that would adequately prepare the generalist elementary teacher for teaching the *Goals* curriculum.

An inspection of the K-6 curriculum reveals an essential framework of four topics: the arithmetic of the real numbers, introductory algebra, informal geometry, and simple applications such as statistics, weights, and measures, with the idea of algebra providing a strong central thread tying together numerical arithmetic and geometry. Curriculum Proposal One is based on acceptance of that framework; each course presents topics which relate directly to the K-6 curriculum, but each treats the mathematics on a level of sophistication and abstraction significantly above that anticipated for K-6. Each course is designed to provide an exciting intellectual experience of significant mathematical merit and to equip teachers with understanding on which they can rely almost daily in the classroom.

It is pertinent here to enlarge on the discussion by which the group came to choose its topics for Course A. If the group followed standard practice they could either have drawn up a course presenting the K-6 topics in greater depth, or simply have chosen any one of several routine college freshman courses. Neither choice seemed satisfactory. The group realized that many students in the elementary education curriculum will have had the minimal mathematics program offered by their high schools and that their previous exposure to mathematics may have left some of them with a dislike or even a fear of the subject.

What the group wanted to find was a topic in mathematics that would be quite new to the prospective teachers and would demand of them only rudimentary knowledge of mathematics, but would offer good prospects of enjoyment and help to reveal similarities of mathematical structure with the topics in the K-6 curriculum. A review of arithmetic or geometry in the same context in which the prospective teachers had studied them in high school would not accomplish the group's aims. There would be nothing really new or exciting in a rehash of old material.

There are many elementary topics in mathematics that begin with ideas new to the prospective teachers but require little background. The important consideration was that the topic chosen should bear on the K-6 curriculum to be taught and should help to make the student a better teacher. The main topics of K-6, of course, would be arithmetic and geometry, which would seem to bring the search back to its starting point. The group wanted a topic that would use elementary arithmetic and apply to geometry; the subject that came to mind was matrices.

In picking matrices for Course A the group was influenced by some of the new British materials, which are notable for the simplicity and lucidity with which they introduce the topic. Since the early stages of the study of matrices involve only the most elementary knowledge of arithmetic, the prospective teachers, while being introduced to a new field of mathematics, would at the same time have an opportunity to review the rules of arithmetic in a context in which the basic laws (commutative, associative, distributive, etc.) have greater interest. The new books, both American and British,* give many interesting, easily understood applications dealing with inventories, coding, transformations, etc., which might be expected to attract the students.

By developing an algebra with properties different from those of the ordinary number system, the student would be in a position to compare different systems and appreciate the significance of the rules they will teach. As the course progresses, matrices are related to transformations of the plane and thus used to study geometry in the context of motions. Further, the notion of angle is studied in connection with rotation matrices, offering a natural introduction to a certain amount of elementary trigonometry.

The presentation of matrices and geometry from this point of view should give the students a fresh approach to the subjects they have studied in high school. An inspection of the K-6 curriculum as elaborated by the group reveals many points of contact with the material of the proposed course in matrices and motion geometry. Some of these points of contact are mentioned in the comments of Chapter V in Part Two, where the detailed description of the course is given. In view of the considerable amount of other work on curricula done by the 1966 Conference, this rather extended discussion of the choice of a single topic may seem excessive, but the assignment of matrices as the first topic in college mathematics for prospective teachers is unusual enough to warrant some explanation. The remainder of the college work of Proposal One will be found in the following brief description of the courses for the entire suggested sequence.

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* For example, such material can be found in the books of Geoffrey Matthews cited earlier, in *The Mathematics of Matrices* by Philip J. Davis (Blaisdell, 1965), and in *Introduction to Matrices and Vectors* by Jacob T. Schwartz (McGraw-Hill, 1961). The SMSG program has material on matrices and their applications appearing in several places.

Course A presents a treatment of geometry from the point of view of affine transformations and their invariants. For an efficient approach to these transformations, there is introduced a small amount of linear algebra, in the form of matrix algebra, for which a number of other applications are also provided. The estimate of one year for this course is approximate, and classroom experimentation will be necessary to determine more closely what can be covered. Following is a brief partial list of topics covered:

Analytic geometry (lines and their properties, circles, triangles, inequalities, convexity).
Matrices and their operations (up to 3 x 3).
Group properties of matrix multiplication (emphasis on 2 x 2).
Systems of linear equations.
Study of certain linear transformations of the plane, using matrices (enlargements, translations, reflections, shears, etc.)
Sensed angles, sine and cosine, measurement of angles.
Elements of vectors (via Cartesian coordinates).
Composition of transformations and the associated matrix multiplication.
Transformations of equations, with application to the development of ellipses and their equations from distortions of circles.

Course B undertakes to illustrate the close ties between abstract algebra and the arithmetic of numbers. The algebraic structure emerges from a study of polynomial functions, which also allows an introduction to the formal differential calculus of polynomials. It is then brought into the open in a discussion of the algebra of integral domains and Euclidean domains, and finally driven home with an introduction to the theory of numbers, in which frequent references are made to parallels with the earlier polynomial algebra.

Functions (concept, examples, representation, operations with, inverses).
Linear functions (graph and slope).
Quadratic functions (examples and motivation, graphs, zeros).
Quadratic formula.
Polynomial functions and polynomial algebra.

Integral domains.
Elementary number theory.

Course C is devoted to applications of mathematics to science and society and to certain mathematical topics which seem particularly amenable to an approach through real problems. In addition to as many examples as possible and to some exposition of the nature and use of computers in the application of mathematics, Course C contains the following mathematical topics:

Systems of linear equations and optimization problems.
Intuitive calculus.
Probability and statistics.
Logic (if time permits).

SECTION 3

Proposal Two for the Training of Generalists

The second group, concerned with the development of basic college mathematics curricula for prospective elementary school teachers, preferred to proceed with a different emphasis and set of priorities. Their work started with a viewpoint common to several of the early internal committee reports and developed to the culmination in this proposed curriculum. What follows is a brief summary and interpretation of the line of thought.

Primary Objectives

The chief deterrent to the realization of any intellectually more ambitious elementary mathematics program is not the teachers' lack of detailed specialized knowledge but the prevalence of crippling attitudes: uncertainty, fear, and hostility. It follows that in constructing the college program first priority should go not to piling up courses in mathematics, but to choosing courses that simultaneously will rekindle the students' interest and build confidence by developing power to cope with mathematical ideas. Furthermore, relevance to the anticipated K-6 curriculum should not be the sole criterion; whether a course opens the possibility of developing or promoting desirable intellectual characteristics in the prospective teacher is of at least equal importance.

Length and Scope

No basis for estimating the necessary amount and depth of mathematical experience is apparent. Since the elementary school curriculum should not and likely will not be static, any actual or hypothetical K-6 syllabus provides only a rough measure of the topical adequacy of a college program. Such a program is successful if it stimulates and supports the teachers' continuing interest and mathematical growth; otherwise it is not. For students whose background in school mathematics is restricted to SMSG Grade 10 or the equivalent, it seems unlikely that a generally satisfactory college program can be managed in fewer than four semester-length courses.

The considerations noted in the foregoing reduce the dependence of college program content on any K-6 curriculum content. It is not crucial, for example, that every K-6 topic in the *Goals* curriculum be repeated in the teacher preparation curriculum. In fact, it is not the recommended topics which give the *Goals* curriculum its special flavor, but a certain attitude toward mathematics in general which is clearly implied by that report. The differences between the *Goals* proposals and the present day K-6 curriculum do not show up as clearly in inspection of their respective tables of contents as they do in comparison of the secondary school curricula into which they logically lead. What is needed, therefore, in the basic preparation of elementary school teachers, now and for some time, is experience providing a mature perspective in the mainstreams of elementary mathematics: the arithmetic of the subsystems of the real numbers and the geometry of the line and plane.

The real number system is important for mathematics not primarily for its intrinsic interest and applicability, but for the many complex and useful structures of higher mathematics which it supports. The second group felt that some introduction to the powerful ideas which evolved in the 18th and 19th centuries from the real number system was an essential part of the training of an elementary school teacher. Those ideas are taken from vector geometry, intuitive calculus, and elementary probability theory. With no more than the bare essentials of these subjects, students can gain an understanding and appreciation of far more interesting mathematical ideas and meaningful applications than a study of the properties of the real numbers alone can possibly give.

Instructional Principles

The preceding remarks suggest not only the natural content of a college mathematics program for elementary school teachers but the appropriate organization and presentation as well. The content of an appropriate program is not, however, uniquely determined. As a basis from which the development of a program can experimentally proceed, two four-semester course programs have been sketched.

The courses in these programs consist of blocks or units whose lengths vary from three to eight weeks. The arrangement of units into courses is based on their very modest degree of conceptual dependency and the principle of including each major subject area in the first two semesters. The use of relatively short, relatively independent units offers several advantages: (1) flexibility in the development of actual courses, (2) repeated fresh starts for students who flounder in particular topics, (3) repeated pressure on the professor, at the beginning of each unit, to connect the mathematical structures with the questions, problems, and situations in which they originate and to which they apply, and (4) exhibition of both the unity and variety of mathematics for students who probably have not yet developed a capacity for the sustained, systematic pursuit of formal mathematics.

Clearly, the style in which courses are conducted has a special impact on students who lack mathematical commitment and maturity. Probably it is especially important that the prospective elementary school teacher be taught by the method his professor hopes will be used in the schools. The challenge of teaching college courses like the ones suggested here is great. One of the most difficult problems is the selection or creation of suitable introductory problems and significant but manageable applications for the units. It should not be inferred, however, that the pedagogy should be restricted to a "discovery" method. Nor is all introductory material equally appropriate. The "fun with mathematics" approach is a tricky tool which has some use as an entrée, but overuse tends to depreciate mathematics and patronize the student.

The unit organization of the curriculum virtually prohibits reliance on the definitional method and its concomitant complex theorem-tree. Rather, the suggested courses proceed from the position that the integers, the rationals, the reals, and the plane exist, fully equipped and more or less familiar. They are indeed to be analyzed as well as applied, but the analysis is to come primarily from reflection after generalizations or comparisons or unanswered questions have raised specific issues, and is not to come by a systematic, formal recapitulation or by means of definitional construction from minimal axiom schemes.

The preceding statement is not intended to advocate a slovenly attitude toward language or argument. In those matters there is no substitute for the good judgment of the professor on questions of accuracy, completeness, formality, and emphasis. The courses outlined are intended as suggestions to encourage mathematicians who have this quality of judgment to extend their concern to the collegiate mathematics program of elementary school teachers and to develop, through trial, effective courses.

OUTLINE OF SAMPLE CURRICULA FOR PROPOSAL TWO

Sample Programs

Course I	A.	*Number Theory.* Divisibility, primes, factorization, congruences, residue-class rings, diophantine equations. (16 class hours)*
	B.	*Vectors in the Line and Plane.* Algebra of displacements, forces and velocities, vector methods in geometry. (8 hours)
	C.	*Transformations and Functions.* Plane to plane functions (e.g., rotations, reflections, projections), general properties of functions, cardinality, common examples of rotations and functions, graphs of functions, binary operations. (12 hours)
Course II	A.	*The Real Number System.* The real ordered field, integers and induction, units in a number line, rationals, decimal representation and gaps in the rationals, sequences and completeness, inequalities and approximations, area of polygons and circles. (25 hours)
	B.	*Counting Problems and Probability.* Enumeration of subsets and Cartesian products of finite sets, outcomes of finite experiments, sample spaces, relative frequency and probability, random variables, binomial distributions, repeated experiments, independence, median, mean, and variance, statistical applications. (22 hours)
Course III	A.	*Intuitive Differential Calculus.* Slope and velocity, algebraic properties and computation of derivatives, differentials as approximations, extremal problems. (22 hours)
	B.	*Linear Transformations and Matrices.* Algebra of 2 by 2 matrices, matrix of a linear transformation of the plane, isomorphism of matrix and transformation algebras, systems of linear equations in two variables. (15 hours)
Course IV	A.	*Isometries and Symmetry Groups.* Isometries in $\mathbf{E}^2$ as distance preserving transformations. Necessary and sufficient conditions for an isometry to be a reflection, rotation, "glide reflection," or translation. Groups of isometries. Groups of symmetries of simple geometrical objects. (15 hours)
	B.	*Quadratic Forms and Conics.* Images of circles under linear transformations, level sets of quadratic forms, classification of conics, plane sections of conics, images of conics under affine transformations. (10 hours)

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* The number of class hours, of course, are intended only as rough approximations.

C. *Intuitive Integral Calculus.* Polygonal approximation of area of regions, area-reading functions associated with graphs, antidifferentiation and area, computation of antiderivatives, applications in mechanics and geometry. (15 hours)

Alternate Sample Programs

Course I′

A. *Circular Functions and Complex Numbers.* Plane curves and their length, similarity of circles, angles and their measures, trigonometric identities, the complex number field and De Moivre's theorem. (10 class hours)

B. *The Real Number System.* (16 hours, an abbreviated version of Course IIA.)

C. *Counting Problems and Induction.* Counting permutations and combinations, the binomial theorem, application of counting formulas to probability, other inductively derived formulas. (8 hours)

D. *Functions.* (6 hours, similar to Course IC, with examples from that course rather than from geometry.)

Course II′

A. *Vectors in Line, Plane, and Space.* The algebra of vectors, linear independence and dimension, bases and components, scalar and vector products. (20 hours)

B. *Intuitive Differential Calculus.* (20 hours, an abbreviated version of Course IIIA.)

Course III′

A. *Rings and Unique Factorization.* Elementary number theory, focusing on unique prime factorization; simple polynomial algebra. (20 hours)

B. *Computational Matrix Theory and Applications.* Matrices and matrix algebra motivated by problems involving tabulation. Matrices and linear transformations in $\mathbf{R}^n$. Nonsingular matrices and inverses, systems of linear equations, matrix representation, solution by matrix inversion. Numerical matrix inversion. (20 hours)

Course IV′

A. *Intuitive Integral Calculus.* (15 hours as in Course IVC.)

B. *Probability.* Repetitions of an experiment and relative frequencies of outcomes, probability space associated with an experiment, derived probabilities of events, repeated Bernoulli trials, independence and conditional probability, Bayes' law, distribution of a random variable, expectation. (15 hours)

C_1. *Statistics.* An introduction to statistical inference centered around the questions: How can we best describe a population sample? From the sample, what kind of conclusions can be derived? How reliable is the conclusion? How shall we design an experiment so that the sample will lead to more dependable conclusions? (10 hours)

or C_{11}. *Impossibility of Angle Trisection.* Constructible points and numbers, fields of constructible numbers, proof of the impossibility of constructing an angle trisector in general. (10 hours)

SECTION 4

Comparison of the Two Proposals

The difference, apart from order, in the actual contents of the two proposals is not startling. The principal difference is, of course, that Proposal One offers a systematic, structural approach, while Proposal Two is motivated by a search for a collection of mathematically coherent topics, each fairly complete in itself, which will kindle interest and promote the teacher's continuing mathematical growth. The reader will note, however, that *within* the topics covered in Proposal Two, the presentation is usually made in a systematic fashion.

A topic occurring in one proposal, but noticeably absent from the other, is trigonometry. There is some trigonometry in Proposal One, mainly the concept and measurement of angles and the use of trigonometric functions in connection with rotations in the plane, but there is none in Proposal Two. (The alternate version of Proposal Two contains some fairly traditional trigonometry, including complex numbers and De Moivre's theorem.) Proposal One is somewhat heavier on geometry than Proposal Two; the entire content of Course A is basically geometry, with other topics, such as linear algebra, introduced as tools in geometry. Though different in manner of introduction, the amounts of linear algebra in the two programs are about the same. Proposal Two contains more calculus, more systematically developed. The amounts of number theory and algebra, and of probability theory, are very nearly the same in the two proposals.

In spite of what was said earlier about the structural nature of Proposal One, it is clear that Course C is of a different sort from Courses A and B. In fact, it is rather in the spirit of Proposal Two, consisting, as it does, of three or four quite independent topics, in the presentation of which there exists considerable flexibility. One of the most distinctive features of Course C is an extensive section on linear equations and optimization problems.

SECTION 5

The Training of Specialists

Secondary school mathematics, for which the *Goals* K-6 curriculum is immediate preparation, includes substantial amounts of algebra, geometry, calculus, probability and statistics. The specialist, of all persons, should be able to guide and encourage the brighter children whose questions probe into the next levels. Both groups agreed that the main point of the specialist's program is that after completing a two-year sequence for generalists, the specialist should study some parts of mathematics in greater depth than is possible in the general two-year course. He might follow a program along these lines:

1. Courses providing further work in probability and statistics, computers and computational methods, or in the applications of mathematics.
2. Electives from such subjects as calculus and numerical analysis, abstract algebra, geometry and topology.
3. A project, substantially equivalent to a semester course, consisting of independent work on a mathematical topic not covered in formal courses.

Chapter III The In-Service Environment of Elementary School Teachers

SECTION 1

Introduction

A typical biography of an elementary school teacher would reveal that the subject was a female whose formal mathematical background usually was capped by a Grade 10 mathematics course in secondary school. She perhaps may have taken one or two semesters of mathematics in college, supplemented by a course in methods of teaching arithmetic. Her total undergraduate program would divide into two or three categories: a general educational block consisting of 50–70 hours in the liberal arts departments; a vocational block of 20–30 hours in education and psychology, including several hundred hours in an elementary classroom; and probably a specialization block of 20–30 hours in some discipline or in an interdisciplinary program. Her undergraduate degree would be in education, awarded by the college of education of a state university or, more likely, by an independent state college which had recently dropped "teachers" from its title.

The K-6 teacher normally must spend six hours at school in almost complete charge of a class of twenty to forty pupils. She may or may not have a short break for lunch. In the upper grades a special physical education teacher may exercise her charges several times a week. In some systems a librarian, a music teacher, or an art teacher will take over for brief periods. A central ditto machine is a standard item. The teacher may have access to a movie projector and a television receiver. Her school system may boast an elementary science or mathematics coordinator.

A state or school district syllabus has shaped the K-6 mathematics curriculum. The appropriate volume of a mathematics text series adopted by local or state authorities has been issued to the pupils. If she has the knowledge and inclination and time, the teacher may be able to supplement the official materials with others of her own devising. Affluent schools may have several texts available.

In addition to her own sense of her needs, there are incentives from her salary schedule or legal requirements that will send the elementary teacher into formal courses. These courses may be organized by the school system and taught by a secondary teacher or supervisor, or they may be offered for credit by local colleges. Typically, they meet once or twice a week, in the late afternoon or evening, for a semester. Frequently such a course is given in mathematics at the time a new textbook series or mathematics syllabus is adopted.

That the young female teacher has a very short vocational-life expectancy was already mentioned in Chapter I, Section 4. Within three or four years a large percentage are not teaching. When some return to the classroom two to twenty years later, their teaching certificates remain valid.

Obviously the environment just described is susceptible to improvement. The remainder of this chapter offers some proposals that would better the environment of the elementary school teacher and, in so doing, increase her effectiveness in the classroom and improve her status.

Because of lack of time and, in some cases, because of the way in which the 1966 Conference was constituted, certain important subjects were not considered, even though the participants were well aware of their importance. For example, the Conference was mindful of the possible significance that new research in cognitive psychology might hold for education in the future. For lack of time and expertise, however, it could only point out the probable importance of new psychological knowledge in the field of mathematical learning, and urge those in educational psychology to work with mathematicians concerned with teacher training and to make their findings available to mathematics teachers in the schools.

The members of the Conference were committed to the idea (1) that the mathematical experiences in grades K-6 should grow out of practical problems or natural situations that are meaningful to the students; and (2) that courses for teachers should provide them with a grounding that will make them feel comfortable and familiar with many interrelations between the mathematics they study and other subjects. In both the proposed curricula and some of the essays in Part III an effort was made to include materials relating mathematics to the natural situations in which relevant mathematical problems can be formulated, but there is still much needed in this direction. Those who develop materials based on the report are urged to amplify the efforts considerably.

The members of the 1966 Conference believe, in addition, that the Steering Committee of the 1963 Conference could facilitate progress in the training of the "whole teacher" and advance the efforts being made by discipline-oriented specialists, if it would bring together some of those specialists who are working on innovative elementary school and preschool programs in all subject areas. The mathematicians might be assisted in designing courses to familiarize teachers with the interrelations between mathematics and other subjects of the elementary curriculum.

Also brought to the attention of the 1966 Conference, but not extensively investigated, was an interesting proposal affecting the classroom environment of the elementary school teacher. It was agreed that teaching experience is a potent stimulus for learning — for learning mathematics as well as for learning to teach. In present patterns of schooling, children rarely teach one another, and the prospective teacher usually has no background of small group, informal teaching experience when she undertakes practice teaching in the third or fourth college year. It seemed clear to the participants that provision of regular teaching opportunities for children of all ages should have a number of beneficial consequences, including an improvement in teacher recruitment. In fact, actual classroom situations were described in which sixth grade pupils were selected to assist in teaching first- and second-graders a few hours each week. Such programs sounded interesting and should be encouraged.

SECTION 2

Use of Specialists

One of the durable educational controversies concerns the relative merits of the so-called self-contained classroom and a departmental organization of the elementary school. The currently predominant compromise suggests one single teacher, to whom we refer in this report as a generalist, for basic academic subjects, and brings in specialists for art, music, and physical education. Predictably, there arose in the 1966 Conference the question of the compatibility of the *Goals* and the class with a single teacher. No formal resolution of this issue was reached, the consensus appearing to be that the Conference was in no position to give an answer. It was argued that the limits of the generalist system first need further testing, specifically by increasing the generalist's resources through the supplementary use of specialists.

Because the elementary mathematics specialist is not a common figure in the schools, it may be well to formulate examples of the work he might contribute in the teaching process and the operation of a school.

Although it is important that mathematics teachers at every level participate in the activities of the profession, the elementary specialist has an overriding responsibility to attend professional meetings, to maintain contact with high school, college, and university mathematicians, and to spend summers and periods of sabbatical leave in renewing and deepening mathematical and pedagogical insights.

One kind of specialist might be responsible for working with other teachers in their specific classroom duties. This specialist must recognize that different levels of competence, preparation, and commitment will be found among the teachers in a school, and he must be sufficiently skillful to raise the level of performance and enjoyment of teachers and students. Typically, this specialist would assist new teachers and teachers beginning a new assignment at a different grade level. He would plan courses with a group of teachers, flag particularly significant or difficult points, and teach in regular classes to provide fresh ideas in content and in method of presentation. He might also be asked to give in-service courses.

A second kind of specialist might have a school-wide responsibility for curriculum planning. He would serve as a consultant for all the mathematics teachers, and might have special responsibility to work with gifted children and with slow learners. Students and teachers would always be welcome for talks about anything connected with mathematics. The specialist would have easy access to college or university people for consultation. He might be involved in lectures and workshops about interesting interrelations between the mathematics curriculum and other subjects. He might also give periodic in-service courses for teachers.

A third type of specialist might be expert in testing. He would make a continuing effort to provide help in the preparation of tests which would reflect the goals of the course and at the same time be interesting and easy to grade. He would assist the teachers in the uses and interpretations of standard tests. He might assist teachers interested in working with upgraded classes or classes taught in unconventional ways, and he could be particularly helpful in judging when children in such classes are ready to go on to new work. Another major focus of his assignment might be to participate in experimental programs, such as TV teaching and programmed learning, and to serve as a consultant to the school on their use. He would probably wish to give occasional demonstration classes, and might run programs for parents.

It is not envisioned that specialists must fall into one of the three categories just described. Indeed, any combination of the following list of seventeen possible activities may be appropriate for a specialist, depending on the needs of the school and the training of the specialist.

1. To teach mathematics only, and to a number of different classes.
2. To help run in-service classes for other elementary school teachers.
3. To experiment with improvements in the mathematics curriculum.
4. To evaluate new textbooks and other pedagogic materials for local use.
5. To give demonstrations of mathematics teaching.
6. To understand and participate in new modes of teaching, such as TV teaching and team teaching.
7. To teach mathematics to parents, and to cope with similar public relations problems.
8. To act as supervisor.
9. To plan the curriculum (and to organize the transition if a new curriculum is in order).
10. To act as trouble-shooter in mathematics and pedagogy for elementary school teachers.
11. To give special help to students (especially at both ends of the spectrum).
12. To advise on the relation between mathematics and other disciplines.
13. To plan the grouping, progression, and testing of students.
14. To specialize in some division of subject matter — perhaps by grade, perhaps by area of mathematics, etc.
15. To communicate with mathematicians about school problems and relate mathematical literature to the elementary school.
16. To attend, and put to good use, professional meetings.
17. To moonlight professionally — e.g., by giving a course at a nearby college in methods of teaching arithmetic.

The following general remarks concerning the training and use of specialists were made during the 1966 Conference and seemed appropriate for inclusion in this report:

1. In general, we believe that mathematics specialists should be teachers with a high degree of authority to innovate and make decisions about the program. Administrators should have specialist training as much as possible. The pay of specialists should recognize the additional training, experience, and excellence they may have.
2. It is essential to the training of a supervisor-specialist that he understand the merits of alternative approaches to the same mathematical topic.
3. The capacity for future growth is a key desideratum in the preparation of the elementary mathematics specialist. His mathematics training should be in the image of mathematics training as we know it.
4. The curriculum of the post-bachelor period, as well as the "continuing education" period, is important to us. Mathematics specialists should

have sabbaticals. Appropriate graduate courses and master's degree programs should be available for elementary mathematics specialists in their continuing education.

5. The mathematics specialist can be of real service in respect to classroom films, classroom gadgetry, and the like. The way to approach such devices is to look for those mathematical ideas whose pedagogic difficulty makes visual assistance most desirable. It can be assumed, we think, that when a clearly defined and demonstrable need exists, the appropriate gadget is inventable. A particular problem where this technique might be very appropriate arises in the applications of mathematics and the associated model building. If for instance, it were decided that a mathematical model of heat conduction should be attempted, then a film or demonstration device to give the student a feeling for the phenomenon to be mathematized can probably be made. This kind of mathematical model building is an area in which teachers are typically very poorly prepared. Probability is another area where clever gadgetry can be really useful.

SECTION 3

Intercommunication in the Area of Mathematical Education

In this country and abroad, a good deal of experimental work is being undertaken in elementary mathematical education. Many of these experiments would be strengthened if those involved could draw upon the ideas and experience of others. Through the resulting coordination of effort we would expect a pattern of cumulative progress to develop. At present, unfortunately, information about one project comes to the attention of others, if at all, only by haphazard and often roundabout routes. Poor communication persists, in part, because a great deal of the material describing these experiments, and used in them, is not written for publication but exists in such forms as mimeographed notes. Furthermore, the items that are published are scattered among very diverse journals, which often are devoted only in small part to mathematical education.

Various suggestions for dealing with this problem of communication were offered at the 1966 Conference: a Resources Center for Pre-College Mathematical Education should be established to bring together all significant materials; a comprehensive, indexed, cross-referenced catalogue of such materials should be constructed; a review journal for the field should be published. From those suggestions the following proposal has been distilled, it seems to meet the most urgent needs and yet appears sufficiently modest in scope to be patently practicable.

We propose that there be issued quarterly a publication consisting of reports on experimental work in pre-college mathematical education. Persons all over the world who are engaged in such work would be invited to make such reports, each one severely limited in length — say, to one page — to be reproduced by photo offset or other inexpensive method. Large-scale projects with many ramifications might report each quarter, others only occasionally. Each report could carry an address from which more detailed materials could be obtained as available.

Those charged with administering this project would be assigned the task of drawing up (and revising from time to time) a key for indexing and cross-referencing these reports. Each person submitting a report would be asked to supply the key for classifying his own report and for referring to previously issued reports.

In the initial phase the project might seek to use the key for purposes of preparing a fairly comprehensive listing of extant materials (the data again to be submitted on a voluntary basis by those involved in experimental work).

For proper functioning, this project should be placed under the supervision of an active organization with close attachment to a very broad segment of the mathematical and educational communities. At the same time, it is highly desirable to maintain sponsorship by some international body to ensure active participation on a worldwide basis. It is hoped, therefore, that whoever undertakes such work will seek an arrangement with the International Mathematical Union (IMU) and other such UNESCO-affiliated unions as may be appropriate to join in sponsoring and distributing the quarterly reports.

Since the formulation of this proposal, the attention of the writers has been drawn to the existence of several proposals aimed at providing the services described in this recommendation. The most ambitious of these is a proposal, made at a conference in March 1966, for a Center for Information on World Mathematics at the University of Chicago. That proposal, indeed, goes considerably beyond the present recommendation. For one thing, its activities would not be limited to pre-college mathematical education, but would encompass college and university mathematics as well. It would bring together in the form of a library all significant materials in the field, much as suggested above for the Resources Center. Its publication program would include not only a quarterly journal for listing and reviewing published materials in mathematical education, but also a journal devoted to expositions of varying length of mathematical topics relevant to the teaching of mathematics, reports of current developmental or research undertakings in mathematics education, invited or contributed papers in areas of current interest in mathematics education, and new developments in mathematics education and of group or individual activity. In addition, it would provide an inter-university research facility in comparative or international mathematical education.

An equally important area of communications involves contact of mathematicians with persons more directly involved with teacher training. A specific recommendation for establishing such contacts via regional conferences is given in Chapter IV. Another recommendation involves the setting up of an "Exchange Center" in the Boston area, where considerable rapport between mathematicians and those involved in teacher training has already been established. The basic idea of operation of such a center is made clear in the following draft of a sample letter to be sent from the center and addressed jointly to the chairman of the department of mathematics and the chairman or dean of education of a teacher training institution in the Boston area:

> I am writing to solicit your cooperation in carrying out a project of the Cambridge Conference on School Mathematics (CCSM), which I hope you will feel may be of benefit to your educational program.
>
> The CCSM is an organization of university mathematicians who have sponsored two conferences on precollege mathematical education — one dealing with curricula, and the other with teacher training — and who have engaged in a variety of experimental projects which involve bringing mathematicians in contact with problems of elementary education.
>
> Out of this experience certain very simple facts have become clear to us: One of the most basic is that mathematicians have much to learn about the preparation of elementary teachers and the practice of their profession. We feel that by now we may have something of value to contribute to a program, such as yours, which is devoted to teacher education. Obviously, to bring about such a sharing of experience requires a meeting ground where views may be exchanged.
>
> It is to provide such opportunities for dialogue that we have set up, in the recommendation of our Cambridge Conference on Teacher Training, what we call an Exchange Center. This is simply an arrangement whereby you may extend an invitation to visit your campus to one of a small group of university mathematicians in the Boston area who have expressed an interest in elementary mathematics education, seeking, on the one hand, to identify distinctive features of your program that may be of value to him and which our organization may wish to consider in greater depth or make the subject of our project work, and on the other hand, to convey to you relevant ideas which have been developed in our conferences and projects.
>
> Such visits would, of course, be arranged without cost to your institution. Any inquiries from you will be most welcome.

SECTION 4

Technological Teaching Aids

Teachers and school administrators cannot form a complete picture of the most effective use of their school system without taking account of the many technological aids now becoming available for elementary education. These include films, closed circuit television, equipment for mathematics laboratories (models, mirrors, Cuisenaire® rods, calculators, mathematical games, games of strategy, etc.), teaching machines, and the more sophisticated technique of computer-assisted instruction. When used in combination with well-trained teachers, these aids can heighten the interest of students, increase the pace of learning, and free the teacher from some routine tasks, allowing her to present more imaginative topics and to lead various discussions.

The Use of Films

Members of the 1966 Conference had the opportunity to see approximately forty instructional films. These were of a great variety: for teaching mathematical concepts and facts to school children; for presenting routine and not-so-routine subject matter to teachers (with and without ideas on how to teach them to children); for demonstrating teaching techniques by exhibiting a "master teacher" at work; for presenting more advanced mathematical topics to high school and college students; and for various other purposes. For comparison, there were films concerned with physics and other areas in physical science. There were high budget and low budget films, formal and informal ones, individual films and films produced as units of long sequences.

As might be expected, there was a variety of reactions to these films, but no firm conclusion was reached as to what properties a good film should or should not have. There was, as might also be expected, a consensus that films would become increasingly important in teacher training as well as in many other areas of education. In the discussions of films, a number of questions were raised and comments made, some of which follow.

a. If a film is being made for fairly large-scale use, changes in the cost of production make very little difference in the cost to consumer, this cost being determined largely by cost of film and distribution. Therefore, for such films there is little point in stinting on cost. However, there could be a real need for low cost films, or low cost video-tape recording equipment, for experimental purposes such as allowing a teacher to see himself in action.

b. Although there is little reason, on the basis of economy, to refrain from the use of elaborate equipment and gadgetry in a film, there may be other reasons for restraint. A demonstration employing simple materials — bottle caps, toothpicks, etc. — might be emulated or even improved upon in the classroom, an instructive exercise not possible if elaborate paraphernalia were used.

c. Is there danger that a student teacher, after watching numerous "master teacher" films, might acquire the feeling that the proper thing to do is to copy tricks and mannerisms? Should this distrust of originality not be guarded against by disclaimers in the films themselves?

d. It would be a desirable feature if many (or most) films had an introductory passage to explain what is going to happen and why, and also an epilogue to fill in points which only become relevant during the showing. These passages might also explain the type of audience the film is aimed at, and whether it is supposed to teach mathematics or teach teaching (something not clear in some films seen).

e. Would it be desirable to make films that can be stopped for quesions, comments, and amplification?

f. Questions were raised concerning the wisdom of using long, filmed courses for teachers, employing a straight lecture method. Some felt that this form of instruction always risks the danger of giving the prospective teacher excellent practice in, and toleration of, dullness.

g. Improvement apparently needs to be made in the methods of handling and distributing films so that films and associated written material can be obtained and used together. Surprisingly enough, this simple matter seems to be a real problem; it is often nearly impossible to obtain or even learn of the existence of written material intended for use with films.

Computer-Assisted Instruction

One interesting new technological teaching aid is O. K. Moore's machine called "The Talking Typewriter." This machine has a keyboard of oversized letters on which the child can type. Initially he strikes keys at random. The letter struck appears on a screen while a recorded voice pronounces its name. At a later stage, a letter is called for and the child finds that all the keys on the board are locked — except the correct one. In this way he learns the alphabet and typing. This practice is followed by spelling out simple words, whose letters must be struck in order, and then on to phrases. The child works at the typewriter when he pleases and for as long as he pleases. The environment contains other things for him to do as well. The report indicates that children of four and five routinely learn to read and write in this way and they even prepare class newspapers.

It certainly seems possible to provide something similar for teaching elementary mathematics. As such machines become more sophisticated, the amount of memory and control required will demand the use of a computer. Such machines can provide specialized and individualized instruction to the pupil. This instruction is one-to-one and permits at each instant an individual response, which in turn determines the future course of instruction. This flexibility, keyed to the individual, distinguishes computer-assisted instruction from other current forms of machine-issued instruction, such as films and television.

Despite the potentially brilliant future of computer-assisted instruction, there are obvious dangers involved in a rapid and widespread adoption of these techniques. The programming of a computer to perform such a function requires a great deal of insight, experience, and patience. Poorly programmed material can easily afflict the pupil with frustration and confusion. A second danger lies in the fact that many teachers will be ill-equipped to decide when and how much and what kind of computer-assisted instruction is most appropriate for a class. Because of these problems the 1966 Conference decided against recommending any widespread increase in computer-assisted instruction at this time.

SECTION 5

Participation of Mathematicians

Substantial numbers of mathematicians have accepted the mathematical preparation of teachers as a subject of real concern; indeed, many have participated in teacher-training enterprises. Mainly, however, they have contributed as individuals. The traditional lack of rapport between members of the educational faculties and liberal arts faculties has operated to separate the mathematical community from formal obligations in the training of teachers.

The 1966 Conference hopes that mathematicians and mathematics departments will recognize their stake in the training of teachers and will seek means for more active participation. Some possibilities are cooperation between professors of education and professors of mathematics in curriculum planning, joint mathematics-education professional appointments, and the teaching of courses in the education school by mathematicians.

There is another important way in which mathematicians are encouraged to support elementary education: they can conduct exploratory classes in elementary mathematics in school classrooms and can cooperate directly with teachers. Several mathematicians have taught such classes and have reported on them in some detail to the Conference. University professors are not the only mathematicians who can make valuable contributions in this manner. An experimental program in two Berkeley (California) schools provides a good illustration.

Since 1946 Berkeley has experienced a growth of the Negro population, presently about 20 percent of the entire city and distributed in a typically segregated housing pattern. A large portion lives in a connected area characterized by relative economic depression and patterns of living that do not emphasize the values making for successful school experience. The levels of performance in the schools of this area are far below those found in other parts of the city, although the school plants and dedicated teaching staffs are universally acknowledged to be fully equal in quality. With wide support in the community, the School Board has moved energetically to erase this disparity of performance.

The program, as described to the Conference, was initiated in the fall of 1964 by William Johntz, a high school mathematics teacher. On his own time, he arranged to teach a fourth-grade class for one hour each day in Longfellow School, one of the four later designated as "target schools" in a "compensatory" program founded by the Office of Economic Opportunity (O.E.O.). The subject matter was to be "advanced mathematics"; the regular teacher, who remained in the room while Johntz taught, would cover the normal arithmetic curriculum by herself at some other hour of the day.

Through the work of Robert Davis and others, Johntz had come to believe that students in the age range 6–12 were capable of understanding more "advanced" and abstract mathematics than the usual arithmetical algorithms, which bulk so large in the elementary curriculum. He felt that students in the socio-economically depressed area around Longfellow School would not be at any disadvantage in such studies, compared to middle-class students, because there was almost no direct link between abstract mathematical concepts and the middle-class cultural and linguistic patterns, which in many parts of the normal school curriculum make preschool experience so relevant to success. If students at Longfellow could experience success in a status-laden field such as advanced mathematics, he argued, it would so alter their self-image and their concept of their own relation to society, that their performance in other parts of the school curriculum would be affected favorably. In all these judgments experience proved him correct.

In the middle of that year, 1964-65, after watching Johntz in action, one of the University of California mathematicians persuaded several Ph.D. students to try their hands at teaching in the elementary schools. By the end of 1965 there were ten classes, including all Grades (1–6) at Longfellow and Lincoln Schools, which were taught by math "specialists" for one hour each day. In addition to Johntz and the Ph.D. students, there were a senior mathematics major from the University, two housewives who had been mathematics majors, and one other high school teacher, who had just graduated from the University as a mathematics major. A portion of the teaching was supported by the O.E.O. program, and another portion by the University. During the coming year, under the code name SEED (Special Elementary Education for the Disadvantaged), Johntz' program was to be incorporated in a comprehensive University program of pre-college education for "disadvantaged" students.

In the conduct of these special hours the regular teacher remains in the room while the specialist is teaching. To her is assigned the responsibility for class discipline, which may require occasional admonishment or sending a student out of the room. Attempts have been made to involve her also in the specialized teaching in auxiliary ways, especially by having her observe individual children and confer with the specialist about their response patterns. There is great variation in the degree of interest shown in the "advanced" mathematical material by the regular teachers. Some have tried to learn a good deal themselves; others remain relatively detached. Among the former group, some have initiated discussions with the special teacher as to how they could integrate their teaching of the regular arithmetic curriculum with the ideas of the "advanced" mathematics. So far none of the regular teachers has felt confident enough to try her hand at serving as a specialist. Unfortunately, the administrative problem of obtaining adequate time for the regular teacher to prepare herself for such work seems too difficult to solve within the present school organization.

But whatever the degree of their personal interest in the special mathematical material, all the regular teachers whose classes are involved agree that they have raised their opinion of the ability of many of their students, and that the class as a whole has been much more responsive to the day's teaching.

SECTION 6

Continuing Education

The 1966 Conference held it unlikely that there would again be such a freezing of the mathematics curriculum as took place in the last century and a half. It therefore appears quite hopeless and inappropriate to expect that preservice training could be extensive enough to equip a teacher for a lifetime. That teachers will continue with their education and, in particular, with their mathematical education, is therefore taken for granted. A second justification for emphasizing continuing education is the mutual interaction between teaching and learning, an alternation which strengthens both. This phenomenon is well-known and much exploited at the university level; it can and should be taken advantage of throughout the educational system.

A third argument is that continuing education will permit and encourage mobility and advancement within the school teaching profession, and, in fact, help to define and establish school teaching as a profession. It seems inevitable that various kinds of mathematical specialists will appear in elementary schools before long, in part because the elementary classroom will be the scene of a wider range and greater depth of activity. To assist teachers in becoming specialists or in shifting areas of specialization, the availability of suitable instruction is essential. It should be understood here that what is meant is mobility *within* the school teaching profession, not training to enable a teacher to take an administrative position and cease to be a teacher. In fact, the notion that a school teacher must abandon the classroom in order to gain financial advancement or social stature must be firmly rejected.

Finally, continuing education should be used as a method of enlivening the teacher's educational activities and her attitude toward them, as well as of relieving the dullness of teaching the same material in the same way year after year, with its sometimes deadening effect on classroom atmosphere and accomplishment. Elementary school teaching must be envisaged as a professional career, carrying with it not only the tangible rewards generally associated with a profession, but also intellectual and spiritual growth, which will surely not be there if monotony and boredom are the teacher's daily companions.

It is obviously desirable for educational change and reform to occur in a continuous and controlled manner, without sudden upheavals or revolutionary changes in curriculum, methods, and environment. Therefore, it seems reasonable to picture the continuing education of teachers as taking place in a dynamic but stable context. If revolutions occur, and they well may (the installation of a *Goals* curriculum over a twenty-five year period might be regarded as one), special methods must be devised to handle them.

Courses for the continuing education of elementary teachers could have several aims. The teacher who is already a mathematics specialist should probably take standard college mathematics courses at the upper undergraduate level. For the generalist the preservice specialist courses should be appropriate, either for simple enrichment or to convert the generalist to a specialist.

Many teachers, however, will not profit most from the more-and-deeper-of-the-same type of instruction. Rather, a kind of instruction that takes advantage of the interplay between teaching and learning may serve them best. In their teaching they will encounter problems of subject matter and of pedagogy which will motivate and direct their continuing education. To take advantage of this incentive requires good communication between groups of elementary teachers and college teachers, directors of summer institutes, etc. For example, the participants in a summer institute might well communicate in advance to the director their interests and problems, and arrange a course or seminar built around them. This type of instruction seems to offer a greater promise of success than one which retraces a section of elementary mathematics for a third time and which is not designed specifically for its clientele.

In this connection, the Conference recommends strongly that departments of mathematics in colleges and universities lend sympathetic attention to the teachers' need for continuing education. Such departments are frequently not hospitable to the type of course that teachers need or want, and departments of education, though they might be appropriate sponsors for such courses, usually lack the staff. Some departments or schools of education have accepted this responsibility, provided themselves with appropriate staff, and negotiated administrative hurdles. Others may have to solve the problem through cooperative efforts of a special organization.

It is quite possible that this type of instruction could also cope with changes in mathematical curricula as they occur. Changes can occur simply by being printed in textbooks. Confronted with change, the teacher should be emotionally and mentally prepared for the instruction and assistance she can get from a course, seminar, workshop or institute that has been organized with her problem in mind.

On the other hand, the subject matter and spirit of what the teacher learns should not be tied too rigidly to immediate application in the classroom. A certain amount of freedom to rove is essential if continuing improvements, changes, and reforms in education are to come from the teachers themselves, from their ideas, their scholarly activities, their experiments, and their organizations. Furthermore, if changes originate in the teachers' intellectual growth they are more likely to be of the desirable gradual evolutionary sort than those introduced from the outside. Changes imposed from outside tend to be abrupt.

Of great importance, too, in this phase of a teacher's career is self-education, whether by day-to-day reading, by experimentation and observation, or by concentrated efforts over long periods such as sabbatical leave. It would appear that success in this direction can come only through the acceptance of a custom which is related to the recognition of teaching as a coherent and respected profession. The Conference strongly believes that teachers need, deserve, and can profitably use the opportunity for this kind of activity.

The mechanics and organization of continuing education will no doubt vary greatly from place to place and time to time, and will depend on available facilities, available funds, and current practices. In particular, transition from the present to a stable but not static stage in which a *Goals*-like curriculum will prevail is doubtlessly a large-scale problem and will require large-scale, mainly *ad hoc*, procedures.

Chapter IV Recommendations

SECTION 1

Preparation of Teaching Materials

The Cambridge Conference on Teacher Training has produced guidelines for the construction of a program for training future elementary school teachers. The proposals include some possible course outlines but no material to carry into the classroom for experimentation. The next step is to produce appropriate material for testing in experimental classrooms, and then finally to revise the material for general distribution.

Those steps can be taken in many different ways. The traditional way is to hope that writers will read the outlines and respond with their own versions of textbooks for commercial publishers. Although there is no quality control in that method and many of the publications will misinterpret the spirit of the Conference report, there may be some books produced that do accomplish some of the aims of the Conference. It was a great disappointment to see how poorly the CUPM recommendations were reflected in most of the textbooks which were written in that way and which professed to implement the CUPM recommendations. We do not want to imply that the individual authors who have ideas for teacher-training texts should be inhibited, but perhaps other efforts should also be sponsored in order to promote some fairly rapid response with some quality control.

A second method that has been tried somewhat successfully operates in the large writing group assembled during a summer, such as the SMSG or the African Workshops. These are crash programs which turn out material rapidly, but with some inevitable unevenness of style. These works usually are rewritten later by one or two people to bring them into a polished form. Even though the first materials were available after one summer, the time taken from the original to the final version may still be several years. By careful choice of the participants and by arranging experimental classes for testing the material in preliminary versions, some control over the quality of the materials produced can be exercised.

A third plan, which combines some of the good aspects of the previous two, is the following. Three mathematicians (or some such number) would be commissioned to spend a summer *individually* developing material for use in a course in the spirit of the recommendations of the Conference. In the following academic year, each would return to his university and try his material on an experimental class of prospective elementary school teachers. The writers would get together the following summer and compare notes on the success (or failure) of their individual courses. The rest of the summer would be spent revising the material and producing a version that could be tested on a larger sample in many colleges. It might be well for this phase of the activity to be conducted at a place where a summer institute for elementary teachers is being held, to permit observation of and consultation with the elementary teachers and their instructors. After this larger test and subsequent revising, a final version would be published for general use.

The Conference believes that this third plan is the one for the Education Development Center to follow. It will involve finding three or more mathematicians willing to spend several summers working on such materials. They should be able to rely on their own universities to provide experimental classes of prospective teachers. Even though the writers may be working on different aspects of the same program, the exchange of ideas and experiences as the project progresses should improve each of the efforts. There would be a definite value in having more than one writer work on the same course.

This approach would allow a certain amount of control both in the choice of writers and the testing of the materials. It would have the advantage of unity of style and ideas, would get the project under way quickly, and would hold promise of turning out a finished product after a few years. If other writers wished to collaborate in the writing, the plan would be flexible enough to permit suitable arrangements. The promise of royalties on the materials produced, plus the summer salaries paid through the Education Development Center, should supplement the natural missionary zeal of the mathematicians sufficiently to induce them to volunteer for this kind of project.

The Conference therefore recommends that the CCSM Steering Committee advise the Education Development Center to set up such a writing program. In addition, it is hoped that the Steering Committee, through the Education Development Center, will continue to encourage individual mathematicians to conduct experiments and write units both for the training of teachers and for school curricula. It is important that those people have the facilities for contact with elementary school teachers and pupils in actual classroom situations. The results of such small scale trials should continue to be made available to those engaged in more global experimentation.

SECTION 2

Communications

A. Meetings

Even after text materials for the proposed courses are available, whether they are to be accepted and used will depend on effective, widespread, and systematic communication of the aims of the report to the faculties of teacher training institutions and school administrators. To achieve this aim it is recommended that meetings of professional mathematical organizations include sessions relating to elementary mathematics education. It is also recommended that regional conferences be held on a regular basis; to them should be invited, as participants, mathematics and education professors who teach courses for elementary teachers, mathematicians from the area who can be expected to be helpful in suggesting how suitable courses may be established and taught along the lines of the CCTT proposals, and administrators from the area who are concerned with mathematics education. CUPM has already established a pattern for regional conferences, and the planning of conferences relating to the work of CCTT should be coordinated with CUPM's pioneering work in this area.

B. Journal

We wish to call attention again to the proposal for a quarterly journal consisting of reports on experimental work in pre-college mathematical education. (See Section 3 of Chapter III.) These brief reports will serve to keep both the curriculum experimenters and the mathematical community informed of the current activities of other groups.

SECTION 3

Continuing Education

General remarks concerning continuing education were given in Section 6 of Chapter III. One of the principal problems in this area is to provide time in which the elementary school teacher may improve her mathematical education to cope better with the changing curriculum. Summer Institutes have been quite valuable and should be continued; the Conference believes, however, that it is crucial that the elementary school teacher be relieved of some class-contact hours each day so that time can be devoted to continuing education. Under such conditions continuing education becomes not a sometime thing but a way of life. In-service courses should be made available, and specialists in various areas should be present for consultation to encourage the teacher and help her to make valuable use of the available time.

The Conference also believes that a sabbatical system should be established in the schools to enable teachers to return to colleges for more intensive and prolonged study.

PART TWO

Details of the Mathematics Curriculum

Chapter V
Details of Proposal One
Chapter VI
Details of Proposal Two

Chapter V | Details of Proposal One

In this chapter we consider in moderate detail the three mathematics courses for the preservice education of prospective elementary school teachers, following the recommendations of Proposal One. Using the same terminology as in Chapter II, we shall designate these courses as Course A, Course B, and Course C.

The discussion in Chapter II gave the rationale for the selection of the subjects in each course. The present chapter gives one version of how the material can be organized for such courses. There are obviously many other ways to assemble such courses, and we hope that this report will stir others to accept the challenge to apply their creative talents to produce texts for an imaginative and stimulating program.

Recall that the group which prepared Proposal One felt that the ties between these courses and the elementary school curriculum should be evident. It is of course impossible to state that the material in Chapter *x* of a college text illuminates units *y*, *z*, . . . , and only these units, in an elementary school curriculum. Any significant mathematical insight affects a teacher's total outlook. But, some connections do stand out clearly, and we have attempted to illustrate them by references in the descriptions of Courses A and B. A reference of the form (2, 17, 20) means that the mathematics in this item has rather obvious connections with the material in units (boxes) 2, 17 and 20 of the elementary curriculum described in Essay 1 of Part III.

COURSE A

Geometry and Linear Algebra

Objectives

The prospective elementary school teacher would, it is hoped, take this course during the freshman or sophomore year in college. The course has objectives of two sorts: on the one hand, to impart specific knowledge (in the sense of concepts, techniques, and so forth), and, on the other hand, to show (primarily by its own example) the general spirit of how one deals with mathematics and with situations involving mathematics.

As the detailed description following will make clear, the "content" objective of the course is to provide the prospective teacher with an adequate background that should enable her (or him) to teach those portions of the K-6 curriculum dealing with geometry and linear algebra.

This emphasis on those topics may sound incongruous in relation to the mathematics of the elementary grades; in fact, however, this recommendation becomes entirely reasonable once one considers the matter in some depth. It is not a flight of fancy to emphasize geometry and linear algebra in such a context. Quite the contrary. Much of the "arithmetic" of the elementary grades is very clearly tied either to geometry or to linear algebra or to both. Some of the traditional "arithmetic" *is* actually linear algebra in false guise. The relation of multiplication to rectangular arrays, the use of the number line as a model for both discrete and continuous arithmetic, and the importance of similarity transformations as a (relatively) concrete aid to a clearer understanding of the abstract notions of ratios and proportion are all instances of how genuine and how deep is the involvement of geometry and linear algebra within the domain of "traditional arithmetic."

The degree of understanding we hope to see teachers achieve should be sufficient to enable them not only to present the material and stimulate interesting discussions, but also to detect misconceptions, and to give satisfactory answers and hints to the bright students who ask penetrating questions. Such an objective implies a great deal, both for the college course itself (i.e., Course A), and for the K-6 school program and its pedagogical style. Just as there is a world of difference between the knowledge required of an actor playing a role in a play and that of the President conducting an unrehearsed press conference, so too there is a world of difference between teaching as "acting out a role" versus teaching as a dialogue with students. What is sought here is clearly the latter.

The spirit of the course is not to dwell directly on the geometry material to be taught to the children, but rather to give meaning to the concepts occurring therein by *using* them and by clarifying them in those situations where the more traditional treatment (very likely taught to the teacher in her own secondary education) is timid, ambiguous, and thereby confusing.

We propose the characterization of mathematics by saying that its essence is ACCURACY achieved through CONCEPTUAL CLARITY in spite of COMPLEXITY. At least, this is what mathematics should mean to the nonspecialist.

Reflection reveals that some courses are more effective than others for transmitting this particular idea. Geometry (when properly taught) is much better at demonstrating the need for refining concepts until their clarity warrants long chains of deductions than, for example, number theory, where the natural definitions are already perfectly clear.

Style of Teaching

Any mathematical discussion will ordinarily contain a formal, official, "for the record" portion, and also an intuitive, informal, "just between you and me" portion. When classroom teachers fail to draw this distinction with some care, one invariable result is confusion in the minds of the students. To invoke an obvious analogy, we might cite arguments between lawyers. In court they are strictly bound by certain rules of procedure and admissibility of evidence; out of court they can let their persuasiveness range as far as decency permits. In both situations, however, they are arguing the law.

In teaching Course A, the teacher should set a good example. He should be careful to signal to the students when he is speaking loosely, intuitively, and off the record. The students soon realize that the course provides not only a logical structure of the material, but also commentary, heuristic and aesthetic remarks, as well as many semi-reliable techniques for arriving at answers which must then be proved before becoming part of the record. (A shift to a bantering form of discourse is effective in signaling that the teacher is speaking loosely.)

For example, if with a straight face the teacher says, "You can think of the point (3, 5) of $\mathbf{R}^2$ either as (3, 5) or as $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$," then he has only himself to blame if the students later fail to distinguish between $\begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$. In general, one should be careful not to suggest that things are identical when they are manifestly different. For example, the equations $x^2 + y^2 = 1$ and $y^2 + x^2 = 1$ are *not* the same, and neither equation *is* a circle. To ignore this caution is at first shocking to the pupils, and eventually leads them to the view that mathematics is a meaningless ritual.

Content

Before looking at the details of Course A, we present a very brief overview of the content in the form of the following list of topics:

Analytic geometry (lines and their properties, circles, triangles, inequalities, convexity).
Matrices and their operations (up to 3 × 3).
Group properties of matrix multiplication (emphasis on 2 × 2).
Systems of linear equations.
Study of certain linear transformations of the plane, using matrices (enlargements, translations, reflections, shears, . . .).
Sensed angles, sine and cosine, measurement of angles.
Elements of vectors (via Cartesian coordinates).
Composition of transformations and the associated matrix multiplication.

Transformations of equations, with application to the development of ellipses and their equations from distortions of circles.

Detailed Description of Topics

1. Analytic Geometry (8, 16, 19, 24)

This is a fairly standard treatment of the topics listed, with one exception. In order to exploit linear algebra efficiently in the "transformation geometry" which is the heart of this course, it is important to have a clear idea of "Cartesian coordinates" as real valued functions defined on $\mathbf{R}^2$.

At the end of the topic we have included a brief treatment of convex sets in the plane and extreme points of linear functions on convex polygonal sets. Such material, with applications, can be found in elementary texts on finite mathematics and linear programming.

a. The Cartesian coordinates x, y as functions defined on $\mathbf{R}^2$. (If p is the point (2, 3), then $x(p) = 2$ and $y(p) = 3$.) In the remainder of this outline of Course A, x and y will always have these meanings. The notation $3x - 4$, for instance, describes the function for which $\{3x - 4\}(p) = 3x(p) - 4$, and (x, y) denotes the function for which $(x, y)(p) = (x(p), y(p))$.

b. A line as the locus of a linear equation. Care should be taken to make clear the relation of a locus to an equation. The locus of the equation $y - mx - b = 0$ is the set of all points p in $\mathbf{R}^2$ such that $\{y - mx - b\}(p) = 0$. (It is permissible — even desirable, even necessary at times — to refer to "the point (2, 3)," so that $\{y - mx - b\}(2, 3) = 3 - 2m - b$. What we wish to avoid is such misleading language as "the point (x, y)" in the context in which "x" and "y" denote the coordinate functions.)

c. Cartesian metric; equations of circles.

d. Intersections of various loci.

e. Angle-bisecting lines; perpendicular lines.

f. Triangles: incidence relations among medians, altitudes, angle bisectors; radius of circumcircle and incircle.

g. Quadrilaterals:

  (*i*) Properties of parallelograms — bisecting diagonals, congruent pairs of opposite sides, opposite angles.

  (*ii*) Properties of trapezoids — diagonals divide in same ratio as lengths of parallel sides.

  (*iii*) Properties of a quadrilateral which are sufficient to insure perpendicular diagonals.

h. Inequalities: graphs, in the Euclidean plane, of sentences of the form $ax + by \leq c$.

i. Convexity: convex sets in the plane; extreme points of linear functions on convex polygonal sets; applications to linear programming.

2. Translations in $\mathbf{R}^2$ (6, 15)

A brief study of translations in $\mathbf{R}^2$ provides an introduction to the notion of geometrical transformation in a simple setting.

a. Illustrate those properties which translations have by virtue of being Euclidean motions (e.g., preserving lengths) as well as special properties such as transforming a segment into the opposite edge of a parallelogram.

b. Introduce notations, "T: $(3, 5) \rightarrow (6, 8)$" and "$T(3, 5) = (6, 8)$."

c. If $T(0, 0) = (a, b)$, then $T(3, 5) = (3 + a, 5 + b)$, etc. Thus T can be described, in terms of the coordinate functions and composition, by $(x, y) \circ T = (x + a, y + b)$ or $(x, y) \circ T^{-1} = (x - a, y - b)$. If we define $\bar{x}(p)$ to be $x(T^{-1}(p))$ and $\bar{y}(p)$ to be $y(T^{-1}(p))$, then we have

$$\bar{x} = x \circ T^{-1} = x - a$$

$$\bar{y} = y \circ T^{-1} = y - b,$$

the traditional "equations of a translation."

3. Matrices and Matrix Operations (21)

The object of this topic is to introduce the notion of a matrix and to motivate the construction of the operations of matrix multiplication, matrix addition, and scalar multiplication of a matrix by a scalar by means of several examples. The examples should be taken from situations which the student can understand readily and should be sufficiently diverse so that the notion of matrix stands out as a unifying idea as well as a useful computational device. The examples should also suggest and anticipate the strong connections between matrices and geometry, which will be made more explicit later in the course.

a. Shopping list (row matrix) times catalog of prices (column matrix) gives total cost. Several shoppers and catalogs lead to $n \times m$ matrices. Combining daily shopping lists to obtain weekly list motivates matrix addition.

b. Intersection of two lines. The system

$$x + 2y = 2$$

$$-3x + 4y = 7$$

can be written

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

Use this example to motivate the inverse of a matrix, showing connections with usual method of elimination.

c. Brief mention of the enlargement transformation T described by $(x, y) \circ T = (ax, by)$, or T: $(x, y) \rightarrow (ax, by)$. Show the connection with the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

4. Algebraic Properties of Matrices (15, 21)

In this and the next two topics we digress to investigate the algebraic structure of sets of matrices. The emphasis is on group properties under multiplication. While this is a purely algebraic segment of the course, the instructor should keep the geometrical destination in view by occasional references to some of the examples in Topic 3.

a. Many examples of addition and multiplication. Show noncommutativity of multiplication and compare constantly with ordinary arithmetic of integers. Use examples where product of two matrices is a "scalar" matrix. From this, ask student to find what will later be called the inverse of a matrix. This anticipates part h.

b. Explain what 0 and 1 (matrices) *should* do. Let students find these matrices.

c. Illustrate associative property with examples such as

$$(a, b, c)\begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

d. Illustrate that $(A + B)C$ is a quicker way of calculating $AC + BC$.

e. Matrices as coding (and decoding) devices. This is a very good topic to motivate *inverses* and *associativity*.

Parts f-j indicate a series of exercises leading to a simple computation of the inverse of a 2 × 2 matrix. They provide a considerable amount of practice with matrix multiplication in an interesting context.

f. Explain the following two matrix transformations:

(*i*) *Transposition*. Call this $A \to A^{T}$. Is $(AB)^{T} = A^{T}B^{T}$?

(*ii*) *Alternation*. This is the transformation $A \to A_{\sim}$, where $A_{\sim}$ is the matrix obtained from A by changing signs according to the scheme

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \cdot & \cdot & \cdot & \dots \end{pmatrix}$$

Show that transposition and alternation commute. Is $(AB)_{\sim} = A_{\sim}B_{\sim}$?

g. Bring in another operation for 2 × 2 matrices, namely by replacing each element by the one in the opposite corner. Call it "square dance." Show that transposition, alternation, and square dance commute.

h. If A is a 2 × 2 matrix, let B be the square dance of $A^{T}_{\sim}$. Then BA is a scalar matrix. Ask student to use this to find inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, etc.

i. Show that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the square dance of A. (Thus square dance is an automorphism.)

j. Show that $\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \multicolumn{4}{c}{\dotfill} \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \multicolumn{4}{c}{\dotfill} \end{pmatrix}$ is $A_{\sim}$.

Thus $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives the matrix B in part h.

Note: Inverses of matrices of higher order cannot be found this way. Higher order inverses should be done later in Topic 6.

5. Subgroups of the Full Linear Group (13, 21)

a. Consider those 2 × 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which have inverses. Call this collection G. Notice that if A belongs to G, so does A^{-1}. Also, if A and B belong to G, so does AB. Thus G is a group.

b. Show that the matrices with $c = 0$ and $ad \neq 0$ form a subgroup.

c. Show that the matrices with $c = 0$, $a = 1$, and $b \neq 0$ form a subgroup.

d. Example of subgroup. Let S be the collection of those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$. Show that they form a subgroup.

e. The complex numbers (in disguise). Show that the matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ form a *field*. (Need not mention 'field' if not wanted.) Show that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, etc.

6. Systems of Linear Equations (13, 17, 18, 21)

This is a fairly standard topic which can be found in many elementary linear algebra texts. The treatment should be restricted to 2 × 2 and 3 × 3 systems.

a. Definition of *system* of linear equations, *solution* ("truth set"), and *equivalent* systems.

b. Reduction of a system to an equivalent diagonalized system; use of matrix representation for the system; row operations; use of the inverse of a matrix.

7. Enlargements and Other Simple Linear Transformations (8, 9, 13, 22)

From this point on, Course A returns to geometry, assisted by matrix algebra. The object is to introduce and treat a representative collection of geometrical objects and ideas through a study of invariants under various affine transformations.

a. Pictorial treatment of *enlargements* as in *School Mathematics Project* (SMP), *Book T*, Chapter 8. Review of material in Topic 3c, but with little emphasis on matrices.

b. Similar pictorial treatment of *enlargements using different moduli* for the two directions (e.g., $(x, y) \rightarrow (ax, by)$), see Topic 3c. Discussion of effect on area of rectangles.

c. *Shears*, $(x, y) \rightarrow (x + cy, y)$, treated pictorially as in SMP, *Book T*. See also D'Arcy Wentworth Thompson's *On Growth and Form*, part of which is included in James R. Newman's *The World of Mathematics*.

d. The matrix of a transformation.

Here the course takes a significant algebraic-analytic turn. The relation between matrices and linear transformations must be established clearly and precisely, but "linear transformation" will generally mean one of the transformations we have discussed. Thus, the problem here is the relation between enlargements and shears and certain matrices.

The point of view which should emerge from this discussion — which will require careful development and motivation — can be summarized by the following:

(*i*) Points are represented by rows: $p \leftrightarrow (3, 5)$.

(*ii*) A column matrix $[p]$ is associated with each point p. Since transposition has been introduced, we may define $[p]$ to be p^T. Thus $[(3, 5)] = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

(*iii*) A matrix $[T]$ is associated with each linear transformation T. In the 2×2 case, the columns of $[T]$ are $[T(1, 0)]$ and $[T(0, 1)]$. If, for example, T is the shear $(x, y) \rightarrow (x + 2y, y)$, then $[T] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(*iv*) If $q = T(p)$, then $[q] = [T(p)] = [T][p] = [T]p^T$. Thus, if p is as in (*ii*) and T is as in (*iii*), the matrix result

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

corresponds to

$$T(3, 5) = (13, 5).$$

Use the point of view established in part d. to discuss reflections in the coordinates.

8. The Euclidean Group of Motions (21, 24)

The purpose of this section is to introduce the important rigid motions and also to suggest the notion of an algebraic system made up of geometrical objects. The word "group" might be introduced formally at this point. No mention of the connection between composition of transformations and products of matrices need be made here unless the observation is made by a student.

a. Introduce Euclidean motions as transformations which preserve the metric. Recall translations; give examples of rotations (defined as Euclidean motions leaving one point fixed).
b. Show that lines go into lines. Thus a Euclidean motion is determined by its effect on a triangle.
c. Discuss product (composition) of two Euclidean motions. Show that these transformations form a group.
d. Show that those Euclidean motions which leave (0, 0) fixed form a subgroup. Call it $O(2)$.
e. If $T \in O(2)$ and $[T] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is its matrix, then $T(1, 0) = (a, b)$ and $T(0, 1) = (c, d)$. Use this to show that $ad - bc = \pm 1$.
f. Define: $T \in O(2)$ *preserves* (*reverses*) *orientation* if and only if $ad - bc = +1(-1)$. Explain geometric meaning, "right hand rules," etc. Show that the orientation-preservers form a subgroup: call it $SO(2)$.

9. Angles, Sines, and Cosines (22, 24)

a. Discuss the *angles* of a triangle and angle in general as two co-originating rays.
b. Define *sensed angle* as angle with one side labeled "initial," the other "terminal."
c. Angles under transformations; congruence under Euclidean motion.
d. Classifications of angles via *standard relocation*, i.e., a translation to put the vertex at (0, 0), and a rotation [element of $SO(2)$] to put the initial side along the positive x-axis.
e. Define sine and cosine as ordinate and abscissa of point of intersection of unit circle and terminal side of standard relocation. Two sensed angles are "the same" if and only if they have the same sine and cosine.

10. The Group SO(2) and Addition of Sensed Angles (16, 21)

This topic will deepen the student's (future teacher's) familiarity with the trigonometric functions while presenting, at the same time, a rather neat application of the matrix representation of a transformation given in Topic 7.

a. Let A be a sensed angle in standard position. There is one $T_A \in SO(2)$ such that T_A sends the positive x-axis into the terminal side of A.
b. Show that $[T_A] = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}$.

c. According to part a. and Topic 9e., T_A (cos B, sin B) = (cos ($A + B$), sin ($A + B$)). But $[T_A(\cos B,\ \sin B)] = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix} \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}$; then formulas for cos ($A + B$) and sin ($A + B$) follow. Many other identities follow; also $A + B = B + A$.

d. Existence of 0 angle. $A + 0 = 0 + A = A$.

e. Existence of $-A$ such that $A + (-A) = 0$.

11. Measurement of Angles (22b)

Here we set up the "degree measure" of a sensed angle. A naive approach might be to define a *1-degree angle* as a sensed angle A such that $A + A + \ldots + A$ (360 times) = 0 ("equals" in the sense of Topic 9a), but $A + A + \ldots + A$ (less tnan 360 times) $\neq 0$. This topic can be discussed for some time, while the students suggest a way out of the difficulties which arise. The consensus might be a scheme like the following.

a. Comparison of angles. Let us call a sensed angle A *common* if $\sin A > 0$. (Common angles do occur in triangles.) If A and B are common, we define $A \leq B$ to mean $\cos A \geq \cos B$.

b. Let R be the standard right angle ($\cos R = 0$, $\sin R = 1$). Begin to define δ (the degree number) as follows:
$\delta(R) = 90$,
if $F + F = R$, $\delta(F) = 45$,
$\delta(F + R) = 135$,
if $T + T = F$, $\delta(T) = 22.5$,
$\delta(R + T) = 112.5$, etc.

The point to get across is that this assigns δ-values to enough angles so that the extra condition, $A \leq B \leftrightarrow \delta(A) \leq \delta(B)$, gives a definition of δ for *all* common angles.

c. Point out that the common notation "cos 45°" really means "$\cos \delta^{-1}(45)$."

d. Point out (but do not necessarily prove) that if A, B, and $A + B$ are all common, then $\delta(A + B) = \delta(A) + \delta(B)$.

e. Emphasize that δ maps common angles *onto* (0, 180).

f. Mention that δ^{-1} has an extension W to all of **R** such that $W(\alpha + \beta) = W(\alpha) + W(\beta)$. We call $W(\alpha)$ *the angle of* α *degrees*. Of course $W(180) = W(-180)$, etc.

12. Vectors (24)

a. Introduce vectors as *ordered pairs of points*. (This is sometimes called a *located* vector. So-called *free* vectors are really translation-invariant vector *fields*. This distinction is considered rather too complicated for this Course.)

b. Introduce Cartesian (*i.e.*, x- and y-) components.

c. Direction angle of a vector as the angle in standard position (relative to the base point of the vector) with the vector lying on the terminal side.

d. Addition of vectors with common base point defined via Cartesian components. Explain parallelogram interpretation.

e. Effect of translation.

f. Effect of rotation about base point. If (a, b) are components of **v**, then components of new vector are obtained from $[T]\begin{pmatrix} a \\ b \end{pmatrix}$.

g. "Dot" product: $(\mathbf{u} \cdot \mathbf{v}) = (c, d)\begin{pmatrix} a \\ b \end{pmatrix} = (c, d)(a, b)^T$.

h. Dot product preserved by rotation because $(c, d)[T]^T|(a, b)[T]^T|^T = (c, d)[T]^T[T](a, b)^T = (c, d)(a, b)^T$, since $[T]^T[T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for rotation.

i. Relate $(\mathbf{u} \cdot \mathbf{v})$ to lengths and cosine of angle.

13. Composition of Transformations (21)

We now relate, finally and firmly, the algebra of matrices and composition of linear transformations. But "linear transformation" still means "one of the transformations we have studied."

a. $T_2 \circ T_1$ defined by $(T_2 \circ T_1)(p) = T_2(T_1(p))$.

b. Show that for all transformations we have discussed (except translations, of course) $[T_2 \circ T_1] = [T_2][T_1]$. (It might also be pointed out that this follows from the fact that $T(p) = ([T]p^T)^T =$ [illegible]$T]^T$, but this may be too sophisticated for the audience.)

c. Explain the significance of T^{-1} for a T which is 1:1. Show that $[T^{-1}] = [T]^{-1}$.

14. Transformations of Equations (18, 22)

a. Let C be a circle or a line in $\mathbf{R}^2$. Let T be a transformation of $\mathbf{R}^2 \to \mathbf{R}^2$. T can be any linear transformation, but it may be best to keep *translations* and *shears* in mind. Then $T(C)$ is the image of C. $T(C)$ is a new circle (or ellipse) or a new line. Problem: How do we obtain the equation for $T(C)$, given the equation of C?

b. Note: we are neither moving the axes, nor the curve. We are merely transforming C into a new curve $T(C)$ which may look nothing like C (*e.g.*, suppose T is a shear). At a point q on $T(C)$, $T^{-1}(x, y) = (\bar{x}, \bar{y})$ gives the coordinates of a point on C, and conversely. Thus, if $f(\bar{x}, \bar{y}) = 0$ is the equation of C, then $f(T^{-1}(x, y)) = 0$ at each q on $T(C)$ and at no other points, so the equation of $T(C)$ is $f \circ T^{-1} = 0$.

c. For *translations*, $T^{-1}(x, y) = (x - k, y - k)$.

d. For linear transformations (recall that these leave (0, 0) fixed),

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = [T]^{-1}\begin{pmatrix}x\\ y\end{pmatrix}.$$

15. Ellipses as Distorted Circles (22)

This topic provides some practice with the ideas in Topic 14, and also a brief introduction to these important conic sections.

a. A *standard* ellipse is the image of the unit circle under a transformation T with $[T] = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$. (See Topics 3 and 7.)

b. The equation of the unit circle is $f(x, y) = x^2 + y^2 - 1 = 0$. Since $[T]^{-1} = \begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}$ and $[T]^{-1}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x/a\\ y/b\end{pmatrix}$, the equation of a standard ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

c. The image of a standard ellipse under any Euclidean motion is called an *ellipse*.

d. Show that if we apply a shear T with matrix $\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}$ to a circle, then the image $T(C)$ is an ellipse because we can find a rotation which transforms $T(C)$ into a standard ellipse.

COURSE B

Algebra and Number Theory

Objectives

The objectives of Course B include the following.

1. The course should deepen and extend the student's understanding of many of the fundamental ideas of arithmetic by a comparative study of the algebra of polynomials and the algebra of the integers.
2. The course should provide the prospective teacher with some of the specific arithmetical information which she will be expected to teach on a lower level of sophistication in Grades K-6.
3. The course should illustrate the process of abstraction by drawing from the study of polynomials and integers more general concepts such as integral domain and unique factorization domain.

4. The course should illustrate the relation of general algebraic structures to arithmetic.

Style

The course begins at an intuitive and informal level. As the course proceeds, methods of proof are introduced so that at the end of the course several important theorems are deduced from accepted facts. One of these methods of proof is mathematical induction, which must enter into the course at an appropriate point.

Content

The course begins with some experience in the origin, interpretation, and representation, by schematic and geometric means, of functions from the real number system to itself. These ideas are then focused on linear and quadratic functions, with special attention given to factorization and rationality of roots of quadratic functions. This leads to questions concerning the roots of and factorization of polynomial functions, with the division algorithm as a principal tool. An algebraic approach to the discussion of tangents to graphs of polynomial functions and a development of Newton's method for root approximations are also suggested.

From the algebra of polynomials and from the algebra of the more familiar integers, the properties of an integral domain are abstracted and exemplified. This provides a new perspective of the division algorithm and of unique factorization in the two basic instances and naturally leads to a closer re-examination of the integer case. As an elaboration and illustration of the fundamentals of number theory, several problems are posed and simple techniques of solution are developed.

Detailed Description of Topics

1. **Functions** (5, 10, 13, 20, 21)

The function concept appears in many places in the K-6 curriculum and is used as a starting place for this course. The student should gain familiarity with the concept of a function from one set to another in a general setting and then specialize to functions from **R** to **R**. A variety of examples should be brought in to show how often the student really encounters the function concept in his daily life. To prepare for the study of polynomials, the basic arithmetical operations on numerical-valued functions have to be studied. The notions of composition of functions and inverse of a function do not enter into the later parts of this course and do not have to be stressed too much. Such discussions of functions can be found in most freshman college textbooks and also in the SMSG *Elementary Functions*.

a. Definition and examples of function from one set to another.
b. Examples of functions from **R** to **R**: constant, linear, polynomial, greatest integer, absolute value, etc.
c. Representation of functions: mapping diagrams, parallel coordinate lines, graphs. (13)
d. Operations on functions from **R** to **R**: addition, multiplication, multiplication by scalar, composition. Identity, inverses, and special properties of composition operation.

2. Linear Functions (13, 16)

We shall later talk about the line which best approximates the graph of a polynomial function at a given point. Thus we want to stress the graph and the slope of a line. The SMSG *Elementary Functions* gives a treatment which should be supplemented by more applications. (13, 16)

a. Definition of linear function, graph, slope, applications. (13, 16)

3. Quadratic Functions (18, 24)

After giving several applied problems (falling body, area, etc.) which involve quadratics, we study quadratics as a model to motivate the study of general polynomials. We graph the quadratic function, locate its zeros, and come to factoring as a useful device. The nature of the roots of a quadratic polynomial can be seen when the quadratic formula is used. Here one can include a brief discussion of the different factorizations that are possible when different coefficient domains are allowed, going from integers, rationals, reals, to complex numbers.

a. Motivating examples such as projectile motion, area problems, etc.
b. Graphing. (24)
c. Locating zeros by factoring. (18)
d. Factoring by completing square; quadratic formula; rational, irrational, and complex roots; discriminant. (18)

4. Polynomial Functions (18, 19, 21, 23)

We want the student to have experience with the integers but we also want to present the material in a new context which may be newer and hence more interesting than a repetition of her earlier experience. We have chosen polynomials as offering such a context. From the algebraic properties of polynomials we shall later abstract to an integral domain and then see that the same properties also hold for the arithmetic of integers, which the student will later be teaching in K-6.

Since the K-6 teacher works in the context of functions, we have decided to restrict the point of view in this treatment to polynomial functions instead of polynomial forms. The polynomial-form point of view emerges in Topic 5 on integral domains, where infinite sequences with finitely many nonzero terms are discussed.

In this early part of the course, things like the division algorithm can be made plausible without a complete proof by induction. The complete proof using induction comes later in the course.

In the last part of this topic, we return to graphing polynomials and we introduce the derivative by first defining the tangent line in terms of contact of a line with a polynomial at a given point. This approach is given in the SMSG *Elementary Functions*. The idea of derivative allows one to locate maxima and minima for polynomials and to solve interesting problems. Newton's method leads to evaluation of zeros.

a. Generally speaking, the entire discussion of polynomials will be motivated by a search for zeros. (18)
b. Operations on polynomial functions with rational coefficients; addition, multiplication, and multiplication by a scalar. (18, 21)
c. Division with remainder; the division algorithm. (14, 18, 21, 23)
d. The remainder theorem: $p(x) = (x - a)q(x) +)p(a)$. (14, 23)
e. Linear factors and zeros. (18)
f. Synthetic division and evaluating polynomials.
g. Simple tests and approximations for rational and integral zeros. (18)
h. Euclidean algorithm, introduced through examples, but proof delayed until Topic 5c; g.c.d. (common divisor of greatest degree), l.c.m. (common multiple of least degree). (14, 23)
i. Mention use of l.c.m. in adding rational functions.
j. Primes in the sets of polynomials with rational coefficients, real coefficients, complex coefficients. Prime factorization. Fundamental theorem of algebra. (23)
k. Return to graphing of polynomial functions with rational coefficients. Order of contact and best linear approximation. Formal derivative. Geometric interpretation. Newton's method for approximating zeros.
l. Higher derivatives. Taylor expansion of a polynomial and simple consequences, such as: $(x - a)^n \mid p(x)$ if and only if $p(a) = p'(a) = p''(a) = \dots = p^{(n-1)}(a) = 0$.

5. Integral Domains (3, 18, 21, 23)

With some familiarity with integers and polynomials, we extract the essential properties and arrive at the axioms for a ring and an integral domain. The infinite sequences with vanishing tails can also be introduced as an example, and this system can be shown to be interpreted as polynomial forms with the indeterminant $x = (0, 1, 0, 0, \dots)$. Here we should bring in other systems familiar to the students and ask whether they are rings or integral domains. Samples are listed in the outline below. This topic is supposed to illustrate how one draws an important abstract structure from familiar examples.

Outline

a. Integral domain structure discovered by investigation of features common to $\mathbf{Z}$ and the polynomial rings just studied. (3, 18, 21, 23)

b. (*Optional*) Abstract definition of a polynomial as an infinite sequence with vanishing tail and special rules of addition and multiplication. Reference to such algorithms as synthetic division and the quadratic formula illustrates that the algebra of polynomial functions is dependent only on the sequences of coefficients and the special arithmetic for those sequences. Thus, the entire subject can be carried out at an abstract level.

c. Contrast integral domain structure with the matrix rings studied in Course A (commutativity, zero divisors) and with $\mathbf{Z}_n$, where n is composite. Compare with $\mathbf{Z}_p$, p prime. (14)

d. Unique prime factorization in an integral domain. Recall polynomials and look ahead to $\mathbf{Z}$. Example of nonunique prime factorization in an integral domain.
$(D = \{a + b\sqrt{-5} \mid a, b \in \mathbf{Z}\}; 9 = 3 \cdot 3 = (2 + \sqrt{-5}) \cdot (2 - \sqrt{-5})$.
Euclidean domain and unique factorization domain (brief mention). (23)

6. Elementary Number Theory (2, 7, 11, 23, 25)

Here we begin a more careful treatment in which we prove more of the theorems. We begin with a discussion of mathematical induction. With induction we can prove theorems such as the fundamental theorem of arithmetic, the division algorithm, and the Euclidean algorithm; then we can point out for what kinds of integral domain the proofs hold. But there are other reasons for introducing a prospective K-6 teacher to mathematical induction. A teacher is very likely to be asked two specific questions by children: "Is there a biggest number?" and "Is 1/0 equal to infinity?" This topic is intended to deal with such questions. As a diversion, Cantor's diagonal process can also be used to discuss the uncountability of the set of infinite decimals. The necessity for discussing statements about countable sets is intended as part of the motivation for Item a. below. The distinction between "mathematical induction" and what is usually called "inductive reasoning" should be pointed out.

Linear diophantine equations can be discussed and perhaps one would want to discuss congruence classes of integers, mod n. The rest of the course can be devoted to a discussion of some of the other famous problems in number theory which may capture the imagination of the class.

a. Induction; its place in axiomatic treatment of Z. Enough exercises to familiarize the student with induction as a technique of proof. Well ordering and infinite descent. (1)
b. Factorization, primes, division algorithm, fundamental theorem of arithmetic, infinitude of primes. (14, 23)
c. Euclidean algorithm, g.c.d., l.c.m. Pause to give proof by induction of division and Euclidean algorithms for polynomials. (14, 23)
d. Linear diophantine equations. (23)
e. Sum and number of divisors.
f. Some introduction to famous solved and unsolved problems, e.g., perfect numbers, twin primes, Pythagorean triples, Fermat's conjecture, etc. (24)

COURSE C

Mathematics, Science, and Society

Objectives

The mathematical education of a prospective elementary school teacher probably cannot be considered adequate unless she gains some reasonable understanding of the place of mathematics in science and in everyday life. We surely do not wish the prospective teacher to regard mathematics as related primarily to space scientists on the one hand, or to grocery lists and retail purchasing on the other hand. The first interpretation is too esoteric to be comprehensible to the prospective teacher, and would be merely exotic, unreal, and meaningless. The second interpretation is far too mundane to interest intelligent students or to give any inkling of the true nature and importance of mathematics.

What we seek, then, is some third vision, at once exciting and reasonably representative, but at the same time "sensible" and comprehensible. If possible, we want to communicate the romance of mathematics, as well as its practicality as a tool *par excellence* for reasoning about the real world.

There is a second reason for wishing to introduce some applications: certain mathematical topics can be learned more effectively in a context of realistic problems drawn from outside of mathematics.

Those two considerations influence the general form of Course C.

In addition, we wish to present in Course C certain important mathematical ideas not included within Courses A and B, and we wish to extend and review some ideas which were included in those previous courses.

Style and Philosophy

The nature of the relationship between mathematics and science has come to be viewed in a new light in recent years, particularly because a great multiplicity of possible mathematical systems has come to be recognized. The central position of mathematics in science seems to arise from the use of mathematics to replace ordinary language in the building of a *mathematical model* of a real situation. The term *science* as used here is to be understood very broadly to encompass not only what is usually understood by the term (i.e., physical and engineering science) but all areas today in which mathematical reasoning has a place, including the social sciences, business and management, life sciences, etc. Course C should emphasize the importance of mathematical model building as much as possible and this should be the central theme. The student should be encouraged to participate in the process of model building, but should also be presented with challenging situations from a variety of sources which have already been preformulated and predigested into mathematical symbols and equations.

Content

The main mathematical subject headings to be covered are the following:*

1. Systems of linear algebraic equations and optimization problems
2. Intuitive calculus
3. Probability and statistics
4. Logic

Of those four topics, the last (namely, logic) is regarded as the least essential for inclusion in Course C, and might be omitted if time limitations were pressing.

One topic not on the list above is nonetheless important and probably deserves inclusion in some way or another — namely, the use of digital computers. Because of the Course's bent toward "real world problem-solving" or orientation toward science, it is natural and essential for such subjects as estimation, analysis of error, checking, and computation to arise throughout in an organic way. The extent to which computing enters, in the sense of high speed electronic computers, presumably will depend upon the knowledge, ability, and desire of the faculty members who teach Course C. The steps from "method of solution" to precise description of algorithm to flow chart to computer program are so close to each other that they should be taken as natural. The amount of actual programming can be kept small or omitted entirely. On the other hand, short concurrent courses in programming might be optional and would give the student valuable additional skill and motivation. Experience in using computers would unquestionably give prospective elementary teachers a valuable tool that could be used in their future professional careers.

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* In fact, the group prepared two alternative (though clearly similar) descriptions of Course C. It may be of interest to reproduce here the alternative breakdown of Course C into major topics:

1. Sets and their applications
2. Notions of infinity, limits, continuity, countability, etc.
3. Questions of logical organization
4. Statistics and probability
5. Computing and the use of computing machines
6. Estimation, checking, and the analysis of errors
7. Practical deductions from physical laws

In the remainder of this section, however, we shall use the breakdown presented in the body of the text, and shall not refer to the breakdown given in this footnote.

Another important component of the course is its applied problems. Some appropriate sources are:

a. Practical arithmetic and geometry
b. Problems of drawing
c. Physical laws: linear, quadratic, inverse square, etc.
d. Vibrations and periodic phenomena
e. Exponential growth and decay
f. Earth, celestial sphere, and heavens
g. Problems of scale and magnification
h. Patterns and formulas

And many others that come readily to mind (including some that will be discussed below in the detailed examination of the contents of Course C).

Detailed Description of Topics

1. Systems of Linear Algebraic Equations and Optimization Problems

The argument is often heard that applications of mathematics are either too trivial or too difficult to be presented in a mathematics course. Linear algebra, however, is one area of mathematics where applications abound which are easy to explain and which are rich in real interest. In the physical sciences one can choose from elementary mechanics (statics, kinematics) and electrical circuits. In the social sciences (perhaps more interesting to the girls) there are endless applications today to "planning problems," from diets to economics.

Course A will have provided some background in matrix algebra. The subject matter here, however, will be systems of linear equations and inequalities, extending through the elements of linear programming. The geometric as well as algebraic aspects should be constantly in mind, from the plane to convex polyhedra. Applications should accompany the introduction of each new topic and idea. The key objectives for the student are:

a. To be able to formulate a problem of this type correctly; i.e., to anticipate the number of independent conditions and to set them down.

b. To understand the conditions under which solutions will exist and be unique.

c. To become familiar with important applications and to learn techniques for teaching this material in the classroom.

d. To be able to apply algorithms intelligently for determining solutions. This work should extend to a comprehension of the elements of computing, including flow charts and rudimentary computer programming.

It is hoped that objective (d) might lead also to discussions with the students concerning logic in mathematics. A formal course on logic is not likely to have much transference value, and in any case the student has to learn about the ritual of a proof — that a definition, for example, means what it says. Such a platitude is not obvious to the naive, as books on science often display, and rational consistency is not regarded by some people as a virtue. Thus, we would mention topics such as: organization of thought via principles of logic; techniques for imparting conviction (using common sense and/or sophistication); and logical grammar, notions of definition and proof. The primary object here is to impart an attitude of mind, rather than a body of facts, and the attitude should be cultivated throughout Courses A, B, and C.

2. Intuitive Calculus

This subject might better be termed *infinite processes* to emphasize its intent and to avoid confusion with the various stereotypes of calculus. The basic intent here is to bring "infinity" into the students' area of discourse. The presentation is to be almost entirely at the intuitive and heuristic level, and time should be spent on developing a "feel" for basic concepts by using methods that might be directly useful in the K-6 classroom. (For example, estimating areas and rates.)

The topics to be studied include:

a. Area and integration. Approximation would be emphasized.

b. Sequences, limits, and continuity.

c. The derivative. Study of motion under constant and variable acceleration.

d. Elementary functions, sin x, cos x, e^x, log x.

e. Simple harmonic motion, growth and decay processes.

Details

a. Estimates of area are incorporated in the K-6 curriculum (see Essay I in Part III), but the emphasis on approximation (which this motivates) is intended here in a much more general spirit. For one thing, we want the students to learn about round-off error (as, for example, in Appendix H of the *Goals* report). Much more, however, is (surprisingly) possible if we look at estimation in a different light; we can actually find first-order approximations to the answers for questions such as:

How much bread is likely to be sold in Chicago each week?

How much rubber flies into the atmosphere from cars in Los Angeles in one day?

If you can see a man's eyes in the distance, how far off is he?

Is a trip to Mars feasible?

In dealing with such questions, the main point of the exercise is essentially to construct a model, and to decide which numerical information can be looked up and which can be estimated. Cultivated to its highest form, this method of attack has enabled some scientists to do excellent work by using shrewd guesses to replace difficult or non-existent mathematics.

Finally, "integration" is intended in the sense of Archimedean-type arguments.

b. Sequences and limits follow naturally from the notions of area arising in Topic (a). As to continuity, it will be necessary to discuss continua as simplified models of granular (atomic, molecular) substances. Centers of gravity, moments, and moments of inertia may be treated for symmetric laminas. Continuous functions may be regarded as those real-valued functions of several variables which have a connected graph. The "inertia" principle (if a continuous function is nonzero at x, its sign remains constant near x) should then be used to draw physical conclusions, concerning, say, the stability or the braking of a car. The "Intermediate Value Theorem" can be related to the task of locating roots of a polynomial equation. Stress should be put on the topological completeness of the number line.

c, d, e. Since these are fairly standard topics, we leave the interpretations to the reader.

3. Probability and Statistics

Our K-6 curriculum requires that prospective elementary school teachers should have some training in statistics. We believe, however, that Course C should contain more statistics, and especially more probability, than appears warranted by the relatively minor role assigned to these topics in the K-6 curriculum.

There is, in fact, no more important component than probability in the general mathematical education of a prospective elementary school teacher, and there is no clearer example of the nature of mathematical models and their interaction with various other intellectual domains. Many of the crucial notions do not require unavailable technical proficiency, but they improve competence with such general apparatus as the set calculus, the handling of functions, and elementary arithmetic.

In order to realize the potential benefits described above, it is necessary to design a course which treats probability spaces primarily as hypothetical abstractions of certain aspects of reasonably familiar situations. This attitude is bound to emphasize the use of probability in statistical inference, principally in the testing of simple hypotheses. As a more specific indication, the following sample, (a)-(d), is offered.

a. *The empirical basis of probability and the creation of probability spaces.*

It is likely that the students involved could profit from designing and performing simple experiments which have unpredictable outcomes and which permit replication. In the CCSM Summer Study Reports of 1965, a sequence of such experiments appropriate to the elementary grades is described. Many of these would be beneath college students but a few long sequences of trials are indicated, since the behavior of relative frequencies and other averages is best observed directly. The data from one large, joint experiment can be kept and subjected to various analyses as the course progresses.

At the same time the subject can be approached by linguistic avenues which pose immediate problems. Examples of the uses of probabilistic statements in common language show that they are used to support arguments and justify courses of action. One object of the course should be to analyze such claims. The field here reaches from the ubiquitous "law of averages" to the rather technical assertions of pollsters and testors. One example of a true probabilistic inference can be selected from each of several fields. Educational testing, medical science, genetics, and accident studies are pertinent sources. The theory can be brought to bear on these issues at several points.

From the properties of relative frequencies a finite probability space can be developed, and its properties and nomenclature can be studied. The problem here is to distinguish the model from the empirical, while at the same time not dissociating them. Often a student so loses sight of the interpretations of formal probabilistic assertions that he is unaffected by surprising or contradictory results. At this point it is convenient to introduce the use of simulated and conceptual experiments using coins, cards, spinners, etc.

b. *Properties of a probability function and their interpretations.* Here simple examples without combinatorial difficulties would be used to associate events in the experimental setting with subsets of the probability space, and to illustrate the problems associated with the conjunction of events. A simple genetic model is an excellent framework for the discussion of conditional probability and for practice in setting up probability spaces.

c. *Use of models in testing hypotheses and inferring properties.* The broad questions here are: How can derived probabilities be used to predict or infer properties in the experimental sphere, and on what ground can it be argued that a probability supports or undermines belief in a particular property? A few examples show the need for systematic counting in the computation of probabilities. Some work in the counting of sequences and subsets can be followed by a discussion of problems involving binomial distributions.

d. *Limit properties of repeated trials.* Some law of large numbers, perhaps only the binomial case, can be derived, with particular attention to the distinction and relations between the theorem and its interpretation. On the other hand, it is not obvious what can or should be done with a central-limit type of theorem, but some attention to the problem is strongly indicated since the teacher will encounter and use normal distributions very often.

4. Logic

As has been indicated earlier, certain aspects of the study of logic are contained in Courses A and B. Time permitting, a more intensive look at logic might be included as the final portion of Course C. Throughout all three courses the students have, in fact, been reasoning and have been considering abstract propositions. One can hope that a growing sophistication toward questions of necessary inference, contradiction, and so on, has accompanied the reasoning.

The major intent of the present material on logic would be to attempt to tie together in one unit much of what has been studied before in a more diffuse way. The emphasis would be on careful interpretation of language, even (where appropriate) in everyday situations. Applications might also include the well-known elementary analysis of switching circuits. It would *not* be presumed that elaborate mathematical apparatus would be used for the development of the system of logic itself, as is done in the more advanced study of mathematical logic (where, for example, *modus ponens* and three axiom schema are used as a foundation for subsequent theorems in logic).

Chapter VI Details of Proposal Two

In this chapter we present outlines, in moderate detail, of the units out of which Courses I-IV and I′-IV′ in Proposal Two are constructed. The numbering of these units corresponds exactly to that in the description of Proposal Two in Section 3 of Chapter II.

UNIT IA

Number Theory (16 hours)

Remarks

Deep and fascinating problems, some of which are still unsolved, lie just below the surface of elementary arithmetic. Some of these should be discussed.

In the beginning, experimental work with numbers which leads to general conjectures needs to be done. The possibilities for making use of such work in K-6 classrooms can be explored.

The choice of topics is flexible. The material selected should allow the assignment of elementary problems and projects which are within the capabilities of the students.

Knowledge of basic properties of the natural numbers is to be assumed. So, for example, no attention should be directed to such matters as the Peano axioms. Also, technical uses of mathematical induction and the well-ordering principle probably should be avoided.

Mathematical Content

Among the possible topics for this unit are the following:

1. Divisibility; prime numbers; existence of infinitely many prime numbers; the twin prime conjecture.
2. Division algorithm; Euclidean algorithm; unique factorization.
3. Congruences; residue classes modulo m; discussion of when $\mathbf{Z}/(m)$ is a field; addition and multiplication tables for $\mathbf{Z}/(m)$.

4. Little Fermat Theorem; simultaneous congruences; Chinese Remainder Theorem.
5. The equation $x^2 + y^2 = z^2$ and its solutions.

Commentary

For motivation this unit might begin with an analysis of a multiplication table. Some numbers in the table do not appear except as products involving 1 — these are primes. A list of primes can be made and the question of whether every number is either prime or a product of primes can be raised.

Questions about factorization should be raised, and problems which lead toward the notion of unique factorization should be assigned. To stress the significance of the uniqueness of factorization of the natural numbers, the lack of uniqueness of factorization in, say, the set of integers of the form $3n + 1$ might be investigated.

Many problems and experiments with numbers can and should be worked into the material developed for topics 3, 4, and 5 above. As an example, one can search for the significance of the final pattern when steps of p units each are taken around a circle marked in n units. This gives rise to the g.c.d. of p and n. As another example, one can investigate the remainders obtained when the numbers $a, 2a, 3a, \ldots$ are divided by 5 and 6 in order to get some idea about when $\mathbf{Z}/(m)$ is a field.

UNIT 18

Vectors in the Line and Plane (8 hours)

Remarks

Considering translations in a line and in a plane gives rise to an algebra of translations. This vector algebra serves as a tool that often yields efficient solutions to problems, many of which might be familiar ones from earlier work in high school.

This work provides an opportunity to demonstrate basic links between elementary algebra and geometry. It also provides an opportunity to make use of a mathematical model to analyze physical phenomena.

Mathematical Content

1. The algebra of translations (or vectors) in a line and in a plane.
2. The algebra of vectors as a mathematical model of composition of forces and composition of velocities.
3. Applications of the algebra of vectors to geometry.

Commentary

The motivation for this unit might come from investigating displacements of a line onto itself. The sum of translations α and β is defined to be the composition of α and β, and the properties of this addition operation on translations is investigated. An algebra of translations in a line is a natural outcome here.

Extend the concept of translation to a plane in such a way that an analogous algebra of translations in the plane is developed. Tracing the "history" of points under translations should bring out the plausibility of commutativity of composition of trahslations in the plane.

Experimental work with a force table, for example, should lead to a recognition of the vector algebra (of translations) as a mathematical model of composition of forces. The physical apparatus used in high school physics laboratories to illustrate resultants of constant speeds in two directions could be used to illustrate that the vector algebra is also a mathematical model of the composition of velocities.

Investigating the consequences of applying translations in a plane to subsets of the plane leads to applications of vector algebra to problems in geometry. This work should give rise to an awareness of the strong links between the algebra of real numbers and Euclidean geometry.

UNIT IC

Functions and Geometrical Transformations (12 hours)

Remarks

There is certainly no best way to introduce the notion of function. Since this concept is a central one in mathematics, it should be introduced at an early stage with many examples and problems. Attention should be paid to translating everyday situations into the language of functions.

Mathematical Content

1. Development of the concept of a function as a mapping from a first set to a second set; equality of functions.
2. Functions from a plane into itself as illustrations of mappings. In particular, translations, rotations, reflections, projections onto a line, shears, similarities, and congruences are easily visualized.
3. Composition of functions; inverses; implications of $f \circ g = I$ for finite and infinite domains.
4. Mappings of a set into itself for finite and for infinite sets; the concepts of injective (one-to-one), surjective (onto), and bijective functions.
5. Relations, numerical functions, graphs.
6. Binary operations as functions; multiplication tables for $\mathbf{Z}_n$.

Commentary

The motivation for this unit might come from examples or problems in everyday situations whose solutions involve setting up functions. The work with translations in the previous unit might provide a natural lead into the concept of mapping a first set onto or into a second set, which can be illustrated graphically.

UNIT IIA

The Real Number System (25 hours)

Remarks

It does not seem appropriate to give a rigorous axiomatic development of the real numbers. Rather, a heuristic approach which gives a sound intuitive feeling for the basic notions involved should be satisfactory.

In this unit a rational number might be viewed as a number which satisfies an equation of the form $ax = b$, with a and b integers ($a \neq 0$). It is definitely *not* intended that rational numbers be defined as equivalence classes of ordered pairs of integers.

A reasonable expectation for this unit is to have the prospective teacher gain some familiarity with the algebraic and order structure of the real numbers and some understanding of the basis for procedures in approximation.

Mathematical Content

1. The operations and relations in a field.
2. Simple ordering in a field; principle of transitivity; the relationships between the field operations and the order relations.
3. The special roles of 0 and 1 in a field; embedding the integers in an ordered field.
4. Mathematical induction; proofs of some simple summation formulas.
5. Choice of a unit length on the number line; division of the unit length into n parts.
6. Rational numbers; density of the rationals.
7. Measures of length: existence of lengths which are not rational, for example, $\sqrt{2}$.
8. Decimal representation; degrees of approximation.
9. Nested intervals; the concept of completeness.
10. Sequences of real numbers; properties of sequences of real numbers; the concept of the limit of a sequence.
11. Inequalities and their manipulation.
12. Measuring the areas of polygonal regions.
13. Measuring the area of a circle by approximating inner areas with inscribed polygons and outer areas with circumscribed polygons.

Commentary

This unit might begin with an investigation of some situation in which multiplication of signed numbers occurs. The students might be asked to formulate situations in which the need for one or more of the four basic arithmetic operations arises.

The relationships between the four basic operations and an order relation can be demonstrated by examples. These relationships can be illustrated graphically by use of the number line. In this regard, the work done with translations in a line in Unit IB might be put to good use.

Using the intuitive notion that a line shouldn't (or doesn't) have "holes," discuss what happens when the side of an isosceles right triangle is used as a unit for measuring the hypotenuse. From such a discussion, the students should get a feeling for the notion that a sequence of nested intervals which get arbitrarily small must determine a unique point.

Activities involving measuring lengths of physical objects and estimating areas of regions, perhaps by tiling, should lead to discussions concerning orders of magnitude in approximations and manipulation of inequalities. In this regard Appendix H of the *Goals* is a useful reference.

UNIT IIB

Counting Problems and Probability (22 hours)

Remarks

One goal of this unit is to introduce the rudiments of probability theory. Another is to analyze and try to understand some of the statistical statements which occur in everyday experience.

It seems reasonable to aim at developing a feeling for the concept of a finite sample space containing equally likely events. Examples of several such sample spaces might be constructed and investigated throughout the unit.

Mathematical Content

1. Combinatorial and enumerative procedures; definition of *n* factorial; counting the number of *r*-membered subsets in an *n*-membered set.
2. The binomial theorem.
3. Random sampling from an urn containing *p* white objects and *q* red objects.
4. Introduction to the elementary language of probability theory — sample space, experiment, outcome, event.
5. Assignment of probabilities to simple events; additive property of probabilities of simple events.
6. Concept of random variable; the binomial distribution.
7. Repeated experiments; concept of independent events; conditional probability.
8. Intuitive meaning of median, mean, and variance; the Tchebycheff Inequality.
9. Remarks on hypothesis testing, estimation, and inference; heuristic discussion of such applications as sampling inspection, quality control, genetics, and population estimates.

Commentary

The introduction of several suitable problems involving counting arrangements can serve to motivate a discussion on various methods of shortening the labor of making such counts. This leads to the notion of the number of permutations of *n* objects (in a row, in a circle, on a necklace).

The distributive law can be employed to explain the coefficients $\binom{n}{r}$ in the binomial expansion of $(a + b)^n$. In discussing the binomial theorem, it is reasonable to develop Pascal's triangle and to relate the number of subsets of an n-membered set to $(1 + 1)^n$.

Develop the notion of a finite sample space with equally likely events by means of concrete examples. Given several such examples, it should agree with one's intuition to define $P(A)$, the probability of an event A, as the number of elements in A divided by the number of elements in the sample space S. In the context of the several concrete examples, the students should be able to develop a feeling for the property that $P(A \cup B) = P(A) + P(B)$, where A and B are disjoint subsets of a given sample space.

In discussing repeated experiments, one should seek to connect the number $P(A)$ with that number between 0 and 1 to which the "frequency of occurrence of event A" eventually settles down. The notion of the law of large numbers might crop up here.

Among the simple applications of conditional probability is an analysis of the difference between sampling with replacement and sampling without replacement.

UNIT IIIA

Intuitive Differential Calculus (22 hours)

Remarks

Perhaps the most obvious and intuitive introduction to the notion of derivative lies in the problem of defining "velocity." This unit begins with a discussion of the position function for an object moving on a line, and then proceeds from the derivative as a velocity to the more general interpretations as limiting slopes of secants and as general rates of change. Only polynomials and the trigonometrical functions are considered.

Mathematical Content

1. Slope of a line and its relation to velocity.
2. Average velocity and its relation to slope of a secant of a curve.

3. Illustrations of item 2 for parabolas.
4. Introduction of velocity as a limit of average velocity.
5. The tangent line as a limit of secants. Identification of derivative with slope of tangent line. Find expression for slope of tangent line at a general point on a parabola and on other simple curves.
6. Rules for derivatives of sums, products, and composites.
7. Derivatives of polynomials and trigonometric functions.
8. Differentials as approximations.
9. Intuitive maximum and minimum problems and examples.

Commentary

One might enter this unit by considering the problem of a freely falling body, working into a quadratic function as a model which fits some reasonable data. Accepting this model, what would be a reasonable definition of the velocity of the object at time t?

UNIT IIIB

Linear Transformations and Matrices (15 hours)

Remarks

This unit focuses on linear transformations of the Cartesian plane. It should have close ties with Unit IC and should stress elementary manipulations and geometric illustrations.

Mathematical Content

1. Definition of a 2×2 matrix.
2. Multiplication of matrices and some study of matrix algebra (including determinants and inverses).
3. The matrix as a linear transformation via the standard basis.
4. The correspondence between matrices and linear transformations. The correspondence is an isomorphism carrying matrix multiplication into composition of linear transformations.
5. Systems of two linear equations in two unknowns.

Commentary

One might introduce this unit by posing the problem: "How can the plane be transformed so that lines go into lines and (0, 0) is fixed?" Such a transformation can be characterized by a 2×2 array (matrix). The unit should be taught so as to illustrate clearly the interrelationships between algebra and geometry.

UNIT IVA

Isometries and Symmetric Groups (15 hours)

Remarks

The purpose of this unit is to provide some additional work with geometry, possibly done from a vector point of view. The subject can be developed either from a synthetic or an analytic point of view; perhaps both approaches should be presented.

Mathematical Content

The Analytic Approach

See Chapter 2, Book T4, *School Mathematics Project.*

The Synthetic Approach

1. In E^2 an isometry is a transformation preserving distance; it is *linear*, *injective*, and *surjective*. Examples: identity, rotations, reflections in a line, translations.
2. Prove synthetically:
    a. An isometry with more than one fixed point is either the identity or a reflection.
    b. An isometry with exactly one fixed point is a rotation.
    c. An isometry with no fixed point is a translation or a "glide reflection."
    d. Any isometry is a product of at most three reflections.
3. The set of isometries is a group. Discuss various subgroups.

4. A *symmetry* of a subset W of $\mathbf{E}^2$ is an isometry of $\mathbf{E}^2$ under which W maps onto itself. *Group* of symmetries of W. Symmetry groups of a square, equilateral triangle, and other simple figures.
5. Symmetry group of some interesting wallpaper designs.
6. Rotational symmetries of lattices; the only possible ones are rotations of 60°, 90°, 120°, 180°.

UNIT IVB

Quadratic Forms and Conics (10 hours)

Remarks

We define and study quadratic forms as functions Q of $\mathbf{R}^2$ into $\mathbf{R}$ defined by

$$Q(x, y) = Ax^2 + Bxy + Cy^2.$$

The theme of this unit is the study of "level sets" $Q^{-1}(r)$ of a quadratic form — more familarly known as *conic sections*. The intrinsic structure of the conics is not considered, nor are foci, asymptotes, etc.

Mathematical Content

1. What happens to a circle with center (0, 0) under a linear transformation of the plane?
2. Define a quadratic form Q in two variables. Study the loci of $Q = 1$, i.e., the set $Q^{-1}(1)$.
3. The discriminant of Q and classification of conic sections. Many pictures and illustrations. Pictorial identification of the conic sections with plane sections of a cone.
4. Quadratic forms under affine transformations.

Commentary

This unit can be introduced very nicely by questions such as: "What is the profile of a spotlight shining against a wall?" or "What happens to the unit circle with center at (0, 0) under a linear or affine transformation?" As a project for this unit the students might apply implicit differentiation (if it has been presented in Unit IIIA) to obtain information about slopes of conics.

If time permits, it would then be possible to discuss some of the focal properties (even though foci and directrices had not been discussed in any detail) and to mention some applications.

UNIT IVC

Intuitive Integral Calculus (15 hours)

Remarks

The integral is introduced intuitively via the usual area problem. It is then related to antiderivatives by the equally familiar intuitive argument. Formulas are derived only for polynomial and trigonometric functions. Several applications are considered as time allows.

Mathematical Content

1. Intuitive idea of area as a limit of areas of polygonal regions. Heuristic definition of area in this way.
2. The function $A(x)$ defined as the area under the graph of f over the interval (a, x) (where f is a positive-valued continuous function).
3. Intuitive proof that $A' = f$, leading to the problem of finding antiderivatives. Non-uniqueness of antiderivatives, and geometrical interpretation.
4. Antiderivatives of power functions, polynomials, trigonometric functions; associated area problems.
5. Applications to displacement of moving objects, centers of gravity, total force (on a dam), work, simple volumes of rotation.

UNIT I'A

Circular Functions and Complex Numbers (10 hours)

Mathematical Content

1. The notion of a curve and its length.
2. The definition of similarity and a proof that any two circles are similar; hence, the ratio of diameter to circumference is constant.
3. Radian measure of an angle. Definition of sine and cosine functions for all real arguments.
4. Identities for trigonometric functions, especially the addition formulas.
5. Complex numbers, their sum and product. Show that division by any nonzero complex number is possible.
6. Polar representation of complex numbers and De Moivre's Theorem.

Commentary

1. We can find the length of a polygonal path by adding the lengths of the linear pieces. What about a circle? Note that the string method doesn't work unless we assume that the string is "inextensible," meaning that its length doesn't change as we wrap it around the circle. Thus, there is an essential circularity in this process of definition. Try other approaches, such as walking around the circle measuring the length of steps, etc.
2. A one-one mapping σ between two sets in a plane or in space is a *similarity* if there is a positive number α such that

$$\alpha d(p, q) = d(\sigma(p), \sigma(q)),$$

where $d(p, q)$ represents the distance between points p and q. Draw the analogy with a "perfect map" (e.g., a "globe" of the world), with the "scale of miles" giving the proportionality (or "stretching") constant. Now show, by constructing the mapping, that circumferences are also related by the stretching constant α, so that the ratio of radius to circumference is independent of size. Note also that any two parabolas are similar. What about ellipses?

3 and 4. This is a fairly standard treatment. The addition formulas follow from the observation that the distance between points

$$(\cos(\alpha + \beta), \sin(\alpha + \beta)) \text{ and } (\cos\alpha, \sin\alpha)$$

depends only on β.

5. First adjoin $\sqrt{-1}$ formally. Later show how this can be done via ordered pairs.
6. A good application of this material is in the development of formulas for $\sin n\theta$ and $\cos n\theta$, in terms of $\sin\theta$ and $\cos\theta$, from De Moivre's Theorem and the binomial theorem.

UNIT I'B

The Real Number System (16 hours)

This is an abbreviated version of Unit IIA. In the shorter time available it will not be possible to cover all the material of Unit IIA, and it is suggested that items 4, 9, 10, 12, 13 might be omitted without compromising the basic objectives.

UNIT I'C

Counting Problems and Induction (8 hours)

Remarks

The theme of the unit is the counting of sets for cases where complete enumeration is unfeasible or inefficient. In all cases considered the sets to be counted possess some sort of symmetry which can be used to obtain formulas.

Mathematical Content

1. Examples of counting problems: "Find the number of different ways of . . ." Emphasize the importance of making clear what is meant by *different*. The problem may be solvable by complete enumeration.
2. Permutations: linear and cyclic; notation: ${}_nP_r$.
3. Factorial notation. Inductive definition of $n!$.
4. Combinations; notation ${}_nC_r$; equality of ${}_nC_r$ and ${}_nC_{n-r}$.
5. Formulas for ${}_nC_r$ in terms of factorials.

6. Binomial theorem. Proof that coefficient of $a^{n-r}b^r$ is ${}_nC_r$ by counting subsets; formal proof by induction.
7. Computation of simple probabilities using counting formulas.
8. Other problems in induction.

Commentary

1. An excellent introductory example is this: In how many "different" ways can the faces of a cube be colored with six colors? The class should be badgered by the teacher until they come up with a definition of "different" for this problem. This will lead to the notion of when two colorings are *equivalent*.
2. Permutations are simpler than the example above, but again note that the difference between linear and cyclic permutations comes from different choices of the "equivalence" concept.
3. Factorials take a long time to compute. Why no short cuts? Compare with computing 3^n, for example.

UNIT I'D

Functions (6 hours)

In this unit some of the concepts and definitions discussed in Unit IC should be introduced. Instead of geometric examples of functions, examples and problems could be drawn from the preceding Unit I'C. Experience in setting up functions from "word problems" should be provided. In the shorter time available here, the essential ideas of parts 1, 3, 5, and 6 of Unit IC might be presented, but in somewhat less depth than intended for Unit IC.

UNIT II'A

Vectors in Line, Plane, and Space (20 hours)

Remarks

The purpose of this unit is to develop the algebra and geometry of vectors, and to use them to study the geometry of points, lines, and planes in the 3-dimensional space of naive physical intuition.

The idea of a vector may be introduced by seeking graphical representation of a displacement of a point, velocity of a point moving along a line, steps and jumps on the number line, force applied at a point of a rigid body, and so on. Here, or at some later time, mention some vector fields, such as the gravitational field of the earth, the velocity field of a moving fluid, the strain field in an elastic body under stress, the (discrete) configuration of velocities of the sun and its planet (and conspicuous satellites), or of cars on an expressway. Contrast with scalars (temperature, elevation, salinity, etc.) which can be measured by a single number on a scale.

Mathematical Content

1. *Vectors.* Ordered pairs of points, or directed segments; length as "magnitude" of an entity.
2. *The origin.* Select a point P and consider (until further notice) only vectors issuing from P.
3. *Multiplication of a vector by a scalar.* Several times as far, as fast, etc., and consideration of same or opposite direction; lead to definition of $r\mathbf{V}$ for any real scalar r and vector $\mathbf{V}$; interpretation on the number line; the zero vector.
4. *Addition of vectors.* Start with swimmer headed directly across a straight river. What is his velocity vector (displacement in unit time)? Or, seek resultant of two forces applied at a point. Definition of $\mathbf{V} + \mathbf{W}$; degenerate cases; abelian group properties (use parallelepiped for associative law); distributivity of multiplication by scalar over *both* additions; triangle law; simple problems — swimmers, oarsmen, airplanes, triangles, parallelograms, etc.
5. *Linear combinations.* The set $\{r\mathbf{V} : -\infty < r < \infty\}$ as the line $\langle \mathbf{V} \rangle$ generated by $\mathbf{V} \neq \mathbf{0}$; compare with the number line; similar definition of $\langle V, W \rangle$, the plane generated by $\mathbf{V}$ and $\mathbf{W}$; definition of linear combination of $\mathbf{V}_1, \ldots, \mathbf{V}_k$ and of $\langle \mathbf{V}_1, \ldots, \mathbf{V}_k \rangle$; the "flats" $\mathbf{U} + \langle \mathbf{V} \rangle$ and $\mathbf{U} + \langle \mathbf{V}, \mathbf{W} \rangle$.

6. *Linear dependence and independence.* What can we write down to express the fact that U and V are not collinear? that U is not in (V, W)? that U, V, and W are not coplanar? Definitions; consequences for the appealing idea of "equating coefficients."
7. *Dimension; basis; components.* Empirical fact: there are sets of three independent vectors, but any set of four vectors is dependent; *basis* as ordered independent triple; components; simple geometric problems; after seeing oblique coordinates a few times, use only orthonormal bases; symmetric equations of line, and parametric equations of lines and planes.
8. *Scalar product.* Definition; properties, metric interpretation of linear forms, equalities, and inequalities; geometric problems; the Schwarz inequality; convexity; optimization problems in the plane.
9. *Vector product.* If class time permits, or as individual project. Geometric applications.

Commentary

2. An alternative is to deal with free vectors, as representations of translations. No matter which attitude prevails, the other should be brought out, along with the isomorphism between the resulting structures. Here is a good place, moreover, to illustrate the idea of an equivalence relation.
6. Much "talk" is needed to familiarize students with the somewhat nonintuitive language used here. It may be helpful to speak of the "trivial" combination (having all coefficients zero); vectors are independent if the only vanishing combination is the trivial one.
7. Here one may recognize and name real linear spaces of dimensions 1, 2, and 3. One may go on to indicate lightly the possibility of linear spaces of higher (even infinite) dimension.

Excellent material and problems may be found in the SMSG *Analytic Geometry*, Chapters 3, 4, and 8.

UNIT II'B

Intuitive Differential Calculus (20 hours)

See Unit IIIA.

UNIT III'A

Rings and Unique Factorization (20 hours)

Remarks

In this unit an introduction to the algebra of rings and integral domains is presented, using, for the most part, the "concrete" examples of the integers and the polynomials with real coefficients. The general idea is to present number theory first and then to take up the algebra of polynomials with real coefficients, continually pointing out the close analogies between those two algebraic structures. The central theme which runs through the unit is divisibility properties and unique factorization.

Mathematical Content

1. Define *factor*, *prime*, and *composite* by looking through a (partial) table for multiplication of positive integers. Discover the uniqueness of prime factorization for small numbers. Pose the general problem.
2. Show example of a set of integers closed under addition and multiplication in which uniqueness of "prime" factorization does not hold. (*Example:* $S = \{3n + 1 : n = 0, 1, 2, \ldots\}$, $400 = 399 + 1$ is in S, and $400 = 10 \cdot 10 \cdot 4 = 25 \cdot 4 \cdot 4$, all factors here being "prime" in S.)
3. Bring out the defining properties of an ordered integral domain by examining the ring $\mathbf{Z}$ of integers.
4. Induction in $\mathbf{Z}^+$; equivalence to well-ordering.
5. Proof of unique factorization directly from well-ordering property.
6. Division algorithm in $\mathbf{Z}$, g.c.d., existence of g.c.d. from the Euclidean algorithm, scales of notation.
7. Obtaining g.c.d. and l.c.m. from prime factorization.
8. Proof of unique factorization theorem from existence of g.c.d.
9. Polynomials and polynomial functions with real coefficients; definition and examples.
10. Division of polynomials, division algorithm, Euclidean algorithm, g.c.d. for polynomials.
11. For polynomial functions: the remainder and factor theorems, and roots of polynomial equations.
12. The fundamental theorem of algebra (refer to Unit I'A).
13. Primes (irreducible polynomials) in the ring of polynomials over the real or complex numbers.

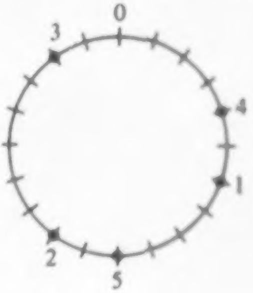
14. As a special project for interested students: Unique factorization in the ring of Gaussian integers. Non-uniqueness of factorization in some rings of quadratic integers.

Commentary

Most of these topics are fairly standard and can be found in most books on number theory and abstract algebra. However, much care will be needed to condense the material and bring it to the appropriate intuitive level for this audience.

One way to motivate some of the material in 1-6 is the following:

Let a circle be divided into $n = 20$ parts. Starting at 0, take steps of length $a = 6$ spaces and mark a heavy dot at each step. The first five are shown in the figure at the right. What is the pattern of dots, if you keep this up indefinitely? Experiment with different n and a. In the case above every other marked point gets dotted eventually. But for other combinations of n and a, some marked points appear never to receive dots. The g.c.d. should begin to appear, as well as a "discovery" that the smallest positive integer of the form $xn - ya$ is the g.c.d. of n and a.



UNIT III'B

Computational Matrix Theory (20 hours)

Mathematical Content

1. Matrices as arrays arising from problems involving tabulation of information. Motivation of form of matrix multiplication in terms of some of these problems.
2. Show that with this multiplication and usual matrix addition, the $n \times n$ matrices form a noncommutative (if $n > 1$) ring with identity.

3. Show that an $n \times n$ matrix $A = (a_{ij})$ defines a linear transformation of $\mathbf{R}^n \to \mathbf{R}^n$:

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} \cdots\cdots a_{1n} & x_1 \\ \cdot \qquad\qquad \cdot & \cdot \\ \cdot \qquad\qquad \cdot & \cdot \\ \cdot \qquad\qquad \cdot & \cdot \\ \cdot \qquad\qquad \cdot & \cdot \\ a_{n1} \cdots\cdots a_{nn} & x_n \end{pmatrix}$$

4. Define A to be nonsingular if the corresponding linear transformation is one-one and *onto*.
5. If the above matrix representation of a linear transformation is written

$$Y = AX$$

then, if A is nonsingular, we can consider the mapping $Y \to X$. Show this is linear. and so has a corresponding matrix, call it A^{-1}. Then

$$X = A^{-1}Y.$$

6. Interpret item 5 in the following way: Solving the system of linear equations $Y = AX$ is equivalent to finding the *inverse matrix* A^{-1} of A. Include numerical work on computing A^{-1}.
7. Discuss permutations and obtain permutation matrix. Show that an $n \times n$ permutation matrix is nonsingular and that its inverse is its transpose. Apply matrix theory to some problems, such as Marriage Rules Problems, in the behavioral sciences.

Commentary

Again, this is fairly standard material which can be found in elementary texts on linear algebra and finite mathematics. See the description of Course A in Proposal One for further details of this nature. Book T4 in the *School Mathematics Project* series contains some interesting material pertaining to item 1.

UNIT IV'A

Intuitive Integral Calculus (15 hours)

See Unit IVC.

UNIT IV'B

Probability (15 hours)

Remarks

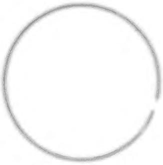
The objectives of this unit are to introduce the concept of probability starting from a frequency approach and to analyze the rudimentary probabilistic concepts.

Mathematical Content

1. Probability space Ω with points ω; events and combinations of events.
2. Existence of probability $P(E)$ for events E in Ω. Laws satisfied by P.
3. Case where Ω is finite; application of combinatorial analysis.
4. Product model for two-part experiments. Repeated Bernoulli trials, binomial distribution.
5. Independence, conditional probability.
6. Bayes' law, applications to genetics.
7. Random variable as a function. Mean, median and variance for finite probability spaces.
8. Idea of a distribution. Random variables with a continuous distribution.
9. The additivity property of expectation: $E(Z_1 + Z_2) = E(Z_1) + E(Z_2)$.
10. Discuss other geometric probabilities.

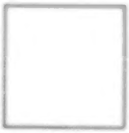
Commentary

It would help to introduce item 1 with a random experiment such as the following. Make copies of the following figures in stiff cardboard:



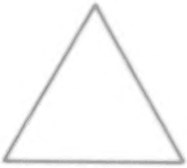
diameter

$\frac{5}{4}$ units



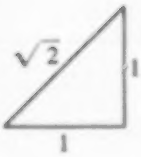
side

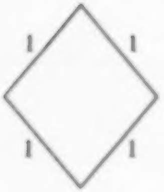
1 unit

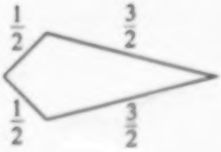


side

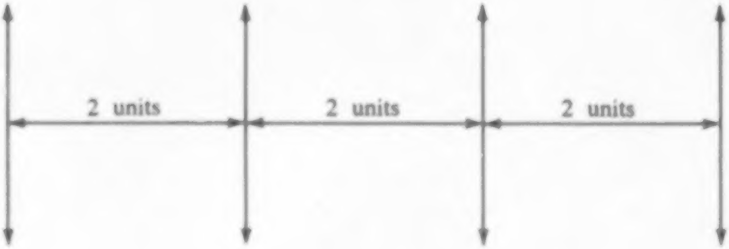
$1\frac{1}{2}$ units







Make a grid of parallel lines two units apart. (In some cases the floor may be good enough.)



Get each person to test a figure at random (discuss best way of insuring this) on the grid and record the frequency with which the figure intersects a line of the grid. If possible obtain at least 500 results for each figure. Review idea of 'frequency settling down' and make a guess at the idealized limiting frequency for each figure. Try to get class to discover what property of figure determines this probability. The results of the experiment can be used for more than one purpose; they should be kept.

From the frequency idea one can deduce that, if an experiment has several mutually exclusive "outcomes," then the frequency of the event that either *A* or *B* happens is the sum of frequencies of *A* and of *B*. Do this first for specific examples like tossing an asymmetric die. Then one can move to the introduction of the probability space. When item 9 is reached the idea and results can be used to obtain a theoretical prediction which can be tested against the results of the experiments above.

In discussing the application of Unit I'C on combinatorial problems make clear that the assumption is a finite Ω in which each point has the same probability. Several examples should be worked out. In item 4 one should tie up the binomial distribution with the binomial theorem. In item 8 one should restrict initially to functions with a finite range for which the integral is available, and item 9 can be tied up with the corresponding idea for integrals. Under item 10, a possible problem would be to compute the probability of rolling a penny at a carnival concession to rest completely inside some square of a mesh.

UNIT IV′C$_1$

Statistics (Optional) (10 hours)

Remarks

This is an introduction to statistical inference. It might begin with a discussion of the making of decisions on the basis of available data. The theory of statistics relates to the collection and analysis of such data. One proceeds first by taking a sample from an unknown population or by performing some random experiment whose probability structure is not understood, and then seeking to reach some conclusions. These are typical questions:

1. How can we best describe the sample?
2. What sort of conclusion can be derived from the sample?
3. How reliable is the conclusion reached?
4. How should we design an experiment so that the sample will lead to conclusions which are more dependable?

Mathematical Content

1. Histograms and frequency polygons. Choice of scales.
2. Samples as probability distributions. Sample median, mean, variance.
3. Inferences from observed success frequency in Bernoulli trials. Confidence intervals, use of samples to test given hypothetical success, frequency p.
4. Normal distributions as approximations to binomial distributions. Use of normal distribution in testing p against sample.
5. Application to situations involving Bernoulli trials.

Commentary

Results of the good experiments of Unit IV′B can be used as the basic example to which items 1-4 above are applied.

Discuss with the student (the prospective elementary school teacher) how she will deal with a child who is worried because he gets 17 heads in 20 tosses of a coin.

Here are other problems of inference which might be analyzed:

Is there evidence to suggest that income at age 35 is likely to be increased as a result of earlier college education?

Is fertilizer A better than fertilizer B?

How do you design a sampling procedure to predict the result of a forthcoming election?

But point out that it is impossible to verify statements like, "The probability that the Democrats will win the next election is 3/5."

UNIT IV'C$_2$

Impossibility of Angle Trisection (10 hours)

Remarks

The problem of making geometrical constructions is fascinating in its own right. In the course of studying why certain constructions cannot be made, the students are introduced to a deeper understanding of fields and polynomial equations. In addition, interesting connections with high school geometry appear.

Mathematical Content

1. Constructible points in the plane. The idea of a construction schema.
2. Algebraic translation of constructibility. The number field generated by $\sqrt{2}$. General number fields. Towers of fields.
3. The angle trisection problem. Outline of solution: If $\cos\theta$ is constructible, then angle θ can be trisected if and only if $\cos(\theta/3)$ is constructible. But if, for instance, $\cos\theta = \frac{1}{2}$, then $\cos(\theta/3)$ satisfies a certain cubic equation with rational coefficients but no rational roots. Such an equation has no constructible roots.

PART THREE

Essays on Some Aspects of Elementary School Teaching

Essay 1
A Mathematics Curriculum for K-6

Essay 2
Teaching Mathematical Reasoning

Essay 3
Some Thoughts on Algorithms for Elementary Mathematics Specialists

Essay 4
Applied Mathematics for Elementary Mathematics Specialists

Essay 5
Experiment in a Model City

Essay 6
What Is a Teacher?

Essay 1 | A Mathematics Curriculum for K-6

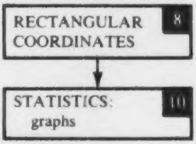
As much to get their own thoughts in order as for any other purpose, the Conference group who prepared Proposal One (Chapter V) devoted their early efforts to constructing a detailed curriculum for Grades K-6 in the spirit of the recommendations of the *Goals* report. They wanted to see, first, whether the generalities of the *Goals* recommendations would still seem reasonable and feasible when reduced to specifics and, second, whether such a detailed sequence would suggest or require the inclusion of particular topics in a college program for the training of teachers. In a broad sense this rather detailed implementation of the *Goals* proposals does not add anything especially new or radically different, but as an example of what might be done along the lines of *Goals* recommendations it is of sufficient interest to be included in this report.

In the process of describing curricula goals there are many dangers, including perhaps the following two especially. In the first place, the description may be excessively ambiguous, or may be seriously misunderstood and misinterpreted; on the other hand, what was intended as a *description* of *a* desirable program may be construed as a *prescription* of the *only* desirable program, thereby denying the importance of alternative proposals. Perhaps the inclusion here of a 1966 reinterpretation of the earlier *Goals* curriculum can help to guard against both dangers. Additional discussion may reduce the chances of serious misunderstandings, while the fact that we have here a reinterpretation and modification of the *Goals* curriculum can serve as an object lesson in the possibility of seeking desirable alternatives and modifications.

The 1966 version presented here was devised for the years of schooling beginning with kindergarten and extending through Grade 6, thereby occupying seven years of the child's life. Adapting this description to a K-8 school or to a lower-middle-upper school organizational pattern should pose no special problems.

We can present the broad outlines of the proposed curriculum in the form of a useful, if cryptic, flow diagram, given on page 99.

The various "boxes" of the diagram are connected by single- or double-headed arrows. Single-headed arrows are intended to indicate a rough logical ordering. For example, the arrow connecting boxes 8 and 10 means that one must know how to make Cartesian graphs before one can use them to display statistical data.



This logical priority will have some limited relation, at least, to pedagogical priority and grade placement of topics, although the priority is not absolute. After all, one can learn about graphs by using them to display statistical data. Such "learning in context" is sometimes especially effective. The double-headed arrows (↔) are intended to connect topics that might be studied in either order, or even simultaneously. Where several topics are listed in the same box, the intent is to suggest that perhaps these topics should be taught more or less simultaneously.

Some of the most important items are deliberately omitted from the diagram, not because they are unimportant, but, on the contrary, because they should be nearly ubiquitous. Chief among these are the concepts of FUNCTION and SET, which should be used throughout the development wherever natural examples and uses occur. By Grade 6 both the words *function* and *set* (and the ideas behind them) should be established firmly and correctly as natural parts of the pupil's mathematical language. The strategy of guessing and checking, or ESTIMATION, should be stressed also and illustrated at all appropriate points. There should be careful attention to the LOGIC in ordinary language and some of the terminology of logical construction might be introduced by the teacher when illustrations arise in classroom discussion.

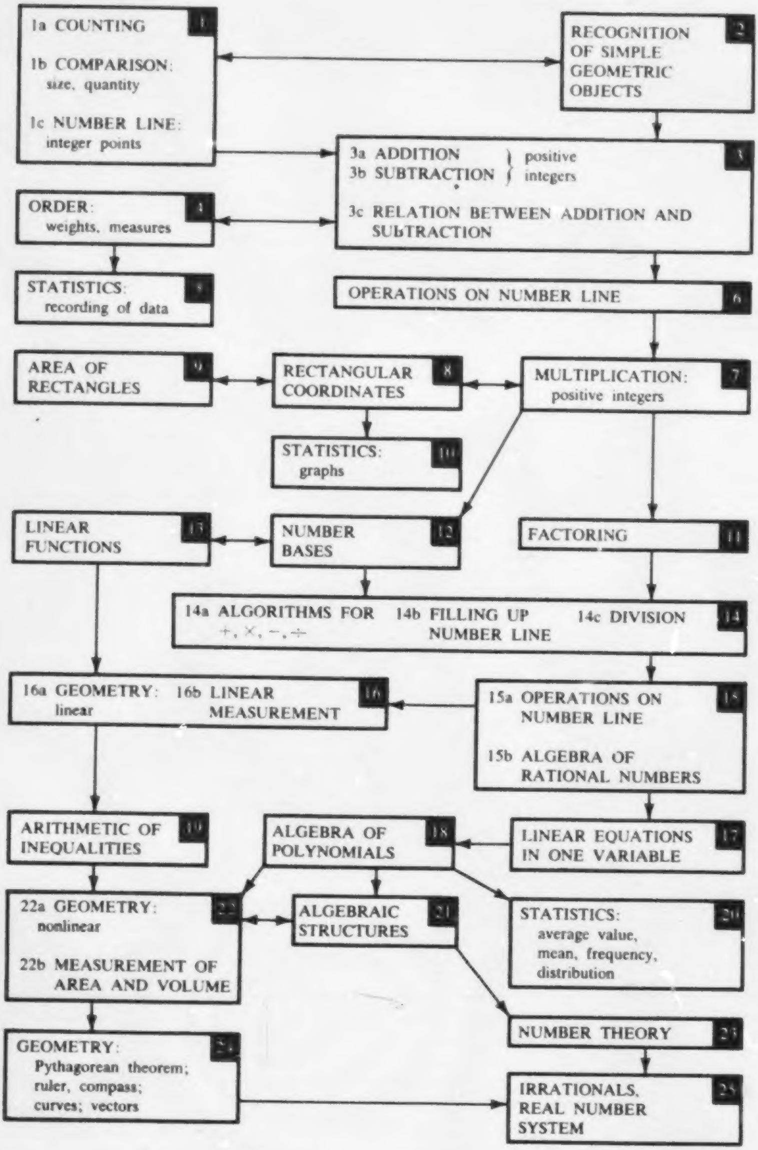
Details and Comments on the Diagram

We present here some explanations of the various topics listed in the curriculum diagram on page 99. The numbers below refer to the numbered boxes in the curriculum diagram. After each title will usually be found a list of words or phrases in parentheses which briefly describe the mathematical content of the unit. Where further amplification is appropriate, these lists are followed by a paragraph of comments.

1. Counting and the (Continuous) Number Line

The transition from the "discrete" arithmetic of the integers to the "continuous" arithmetic of the real number system is commonplace for a mathematician. Nevertheless, the distinctions between these two number systems,

A Curriculum for K-6


1a COUNTING
1b COMPARISON: size, quantity
1c NUMBER LINE: integer points
1
RECOGNITION OF SIMPLE GEOMETRIC OBJECTS
2
3a ADDITION } positive
3b SUBTRACTION } integers
3c RELATION BETWEEN ADDITION AND SUBTRACTION
3
ORDER: weights, measures
4
STATISTICS: recording of data
5
OPERATIONS ON NUMBER LINE
6
AREA OF RECTANGLES
9
RECTANGULAR COORDINATES
8
MULTIPLICATION: positive integers
7
STATISTICS: graphs
10
LINEAR FUNCTIONS
13
NUMBER BASES
12
FACTORING
11
14a ALGORITHMS FOR +, ×, −, ÷
14b FILLING UP NUMBER LINE
14c DIVISION
14
16a GEOMETRY: linear
16b LINEAR MEASUREMENT
16
15a OPERATIONS ON NUMBER LINE
15b ALGEBRA OF RATIONAL NUMBERS
15
ARITHMETIC OF INEQUALITIES
19
ALGEBRA OF POLYNOMIALS
18
LINEAR EQUATIONS IN ONE VARIABLE
17
22a GEOMETRY: nonlinear
22b MEASUREMENT OF AREA AND VOLUME
22
ALGEBRAIC STRUCTURES
21
STATISTICS: average value, mean, frequency, distribution
20
NUMBER THEORY
23
GEOMETRY: Pythagorean theorem; ruler, compass; curves; vectors
24
IRRATIONALS, REAL NUMBER SYSTEM
25

the difference between the properties, uses, and capabilities of these two arithmetics, are not generally understood by the layman. Neither are they understood by most elementary school teachers. Partly as a result of this, the transition from discrete arithmetic to the algebra of the real numbers is a difficult one for the elementary school pupil. In fact, this transition is often delayed until the secondary school years, so late that it is made all the more difficult by the rigid patterns and habits formed in six or more years of elementary school arithmetic.

To ease the transition into "continuous" arithmetic and to avoid, at this stage, the subtleties of constructing numbers, one should introduce the complete number line in the very beginning. That is not to say that the real number system is an object for discussion in Grade K. At first, only certain points are marked and named by numbers. But the entire line is there and the points and segments on it should be discussed freely and manipulated from the beginning. New names are added only as the need arises, but the connection between the arithmetic operations and geometrical constructions and manipulations is always explicit. By the time the child is ready to discuss the real numbers *per se* he finds that, in fact, he has been working in the real number system for a long time.

1a. *Counting.* (Objects in sets, points on a clock dial, steps along a path, etc.)

Counting can be formalized as "pairing off" with labels on the number line (see 1c). The notation $\#A = 7$ might be introduced to mean that when one pairs off the objects in A with the integer points (starting at 1), then 7 is the last number to be paired. The independence of $\#A$ from any particular "pairing" should be illustrated by repeated trials and simple tricks for finding $\#A$ without direct recourse to the number line, and should be introduced at an early stage.

1b. *Comparison.* (Sets, weights, lengths, liquid measure, etc.)

At first, only qualitative judgments of "more," "smaller," and "heavier" are called for, but soon the crude beginnings of measurement can be introduced. Only situations calling for whole numbers are anticipated at this stage. Here, also, comparison is accomplished by direct physical superposition.

1c. *Number line.* (Continuous number line; integer points marked.)

When the number line is introduced, all integer points (both positive and negative) should be labeled. Many everyday examples, such as thermometers and time classes, can be used as motivation.

2. Recognition of Simple Geometric Objects

Points, lines, and segments can be recognized in the physical environment, and these physical objects can be related to idealized abstractions. Simple shapes such as rectangles, triangles, and circles can be treated in similar fashion. Various properties such as symmetry can be considered. Maneuvers with points and segments on the number line can be used to foreshadow the numerical operations soon to be introduced. Simple decompositions and a qualitative introduction to area (following 1b) should be included, as well as empirical verification of translation- and rotation-invariance of area, i.e., "If the shape fits the hole here, it fits it there, also."

3. Addition and Subtraction of Positive Integers

3a. *Addition of positive integers.* (Combining collections of objects and counting their number; the usual language of "$2 + 3 = 5$"; empirical observation that $a + b = b + a$; $a + 0 = a$; $a + (b + c) = (a + b) + c$; and problems like $3 + \square = \triangle$ and $a + 3 = 5$.)

3b. *Subtraction of positive integers.* (Subtraction introduced via splitting sets into two sets (so we consider $a - b$ only when $0 < b < a$); language of "$5 - 3 = 2$"; empirical observation that $a - 0 = a$; $(a - b) - c = a - (b + c)$; problems like $3 - \square = \triangle$; $a - 1 = 2$.)

3c. *Relation between addition and subtraction.* Observe that $a + (b - a) = b$, and note that $a + b = c$ expresses the same idea as $a = c - b$. This will help later to tie together addition and subtraction as inverses of each other.

4. Order

(Weights and measures, introduction of inequality signs, problems like $3 < \square$ and $\square < \triangle$, transitivity.)

5. Statistics

(Geometrical representation of data using bars, simple graphs, size of figures, "pies," etc.)

6. Operations on the Number Line

(Vector approach to addition and subtraction.)

The point of view is now changed to regard the number 3 (say) as the vector (or trip) from 0 to 3 on the number line. Addition now becomes the operation of laying lengths end to end, so that "Go to 3 and count 4 steps forward" is identified with the vector addition "$3 + 4$." Similarly with subtraction. Many of the algebraic properties of addition can be recognized empirically. The addition slide rule and various games with sticks or rods should be introduced. These vector operations will later be generalized from integers to reals. To create a suitable background, the universe could now be widened to allow arbitrary segments from 0 to be added vectorially.

7. Multiplication of Positive Integers

(Positive integers only, introduced as repeated addition, $a \times b = b + b + \ldots + b$ (a times); counting rectangular arrays; $a \times b = b \times a$ by empirical observation; also $a \times (b + c) = (a \times b) + (a \times c)$ empirically; multiplication slide rule; properties of 0 and 1. Note that "counting rectangular arrays" can include arrays of physical objects such as pebbles.)

"Multiplication facts" and systematic strategies for multiplying would be introduced here as a response to the problem of finding efficient methods for counting rectangular arrays. In compiling tables (or in solving problems) use might be made of the distributive law, as in

$$6 \times 32 = 6 \times (30 + 2) = (6 \times 30) + (6 \times 2).$$

At this stage of study the basic tool would probably be the process of counting.

It would also be possible at this stage for the child to get some experience using (and even constructing) logarithmic-scale slide rules.

8. Rectangular Coordinates

(Lattice points only; use of games involving the locating of points; trips in two directions; symmetry.)

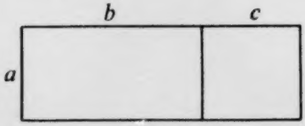
Here many games can be invented to provide practice in locating points by means of rectangular coordinates. The familiar games "Four in a Row" and "Battleship" are examples.

9. Area of Rectangles

(Tiling; area as (vector) product; reconciliation with repeated addition; area of right triangle; a new look at the distributive law.)

Here the connection between area and counting rectangular arrays of (unit) tiles is obvious — hence the connection with multiplication.

A *use* for symmetry may be pointed out in connection with the area formula for a right triangle. One can also begin to observe that a definition of area as a product makes sense for rectangles with sides of arbitrary length. The distributive property can be illustrated from the additivity property of area:



10. Statistics

(Representation of data on coordinate axes; line graphs.)

11. Factoring

(Using integers only; problems such as $12 = \square \times \triangle$, $\square \times 3 = 6$; repeated subtraction; use of multiplication slide rule.)

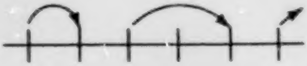
12. Number Bases

(Several bases; game of trading chips of various denominations; abacus to any base; properties of remainder, e.g., $15 = (2 \times 6) + 3 = (7 \times 2) + 1$.)

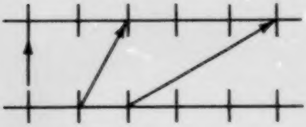
13. Linear Functions

(Functions as correspondences, with several alternative geometrical representations.)

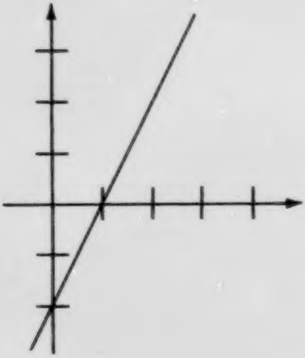
Functions should be represented on one number line,



on parallel lines,



and in the usual way by Cartesian graphs:



The Cartesian graph representation is important for work which will appear later (as well as for use in modern elementary school science programs).

14. Algorithms, Filling up the Line, and Division

14a. *Algorithms for addition, multiplication, subtraction, and division*. (The usual algorithms; integers only; various bases; using geometrical analogies; terminating decimals introduced; use and design of desk calculators.)

14b. *Filling up the number line*. (Integer and rational points; fractions introduced as "new" names; decimals; qualitative discussion of continuum.)

14c. *Division*. (Division as decomposition of sets; remainders; decimals; division as inverse operation of multiplication; fractions — the objects but not the arithmetic.)

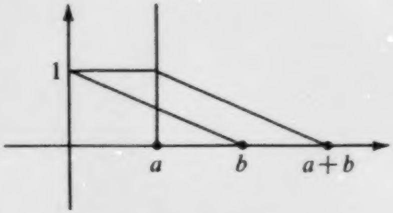
15. Operations on the Number Line; Algebra of Rational Numbers

15a. *Operations on the number line*. (Geometric constructions in the plane leading to the arithmetic of the number line; reconciliation with previous results for positive integers; refinements of mesh.)

This topic requires explanation. Discussion in our group showed that, while multiplication of positive integers arose earlier from problems natural at that stage, the question "Why and how should we define $a \times b$ when a and b are not necessarily positive?" is not natural at this stage. Illustrative situations can be developed for motivating the multiplication of negative numbers (e.g., train traveling on a number line with certain speed in either direction; where is it at some time *before* $t = 0$?), but they usually rest on concepts, like time and speed, which are themselves difficult. The group therefore decided to transform the setting entirely to geometry. The arithmetic will be an arithmetic of segments, or vectors, and the problem is to find ways of defining addition and multiplication by means of natural constructions. Some of this work has been done earlier for addition (go 3 steps this way, then 5 that way).

Since the most natural geometrical construction of multiplication on the number line involves constructions in the plane, it might be wise to set up

Figure 1



one or more methods of accomplishing translations along the number line by a series of projections involving the plane. One such construction is suggested by Figure 1. This can be used to define addition for arbitrary segments in either direction; empirical observation can be used to obtain various properties of addition when a and b are integers. With this experience of "going into the plane," we can now construct $a \times b$, for any positive integer a and any real number b, by plotting the graph of $y = x \cdot b$ for $x = 0, 1, 2, \ldots$. After it is shown (see Figure 2) that this gives $a \times b$ as $b + b + \ldots + b$ (a times) *as before*, it becomes natural to extend the definition of multiplication in two steps:

A. Define $a \times b$ for arbitrary real $a \geq 0$ by connecting points of the discrete graph by segments (Figure 3a).

B. Define $a \times b$ for $a > 0$ by extending the linear graph to the left (Figure 3b). If this diagram is constructed when both a and b are negative reals, we see at once that $ab \geq 0$ (see Figure 3c). Commutativity and associativity are observed empirically; distributivity is immediate from the construction, as are the rules for 0 and 1.

Figure 2

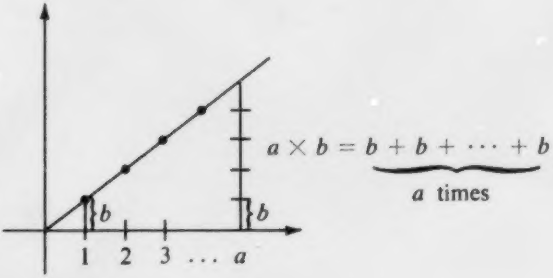
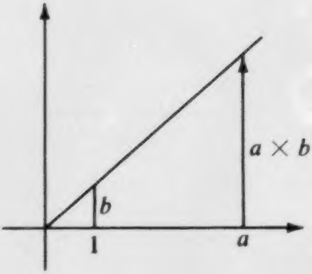

b
b
1 2 3 . . . a
$a \times b = \underbrace{b + b + \cdots + b}_{a \text{ times}}$

Figure 3a



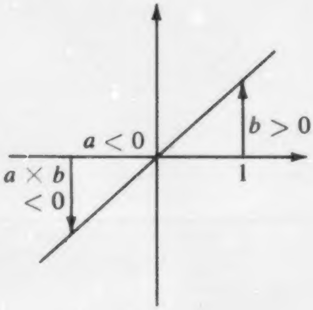


Figure 3b: $b > 0$

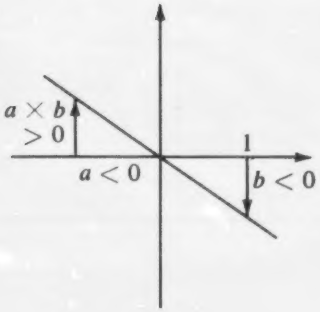


Figure 3c: $b < 0$

15b. *Algebra of the rational numbers.* With the operations of addition and multiplication thus defined for "arbitrary" segments, one may, of course, produce subtraction and division, and proceed to illustrate all the usual algebraic properties. Since the only points on the number line which actually have been labeled are the rational points, this is a good opportunity to construct all the usual algorithms for operating with fractions.

16. Geometry and Linear Measurement

16a. *Geometry.* (Linear properties; mensuration of lengths, perimeters, circumference; similarity, change of scale; experimental determination of π.)

16b. *Linear measurement.* (Weight; length; angle; standard units; estimation and error analysis; work with approximations.)

17. Linear Equations in One Variable

Learning to solve $ax + b = 0$ involves and reinforces the ideas of topics 13-16 and prepares the way for topic 18.

18. Algebra of Polynomials

(Exponents; manipulation of binomials and trinomials; quadratic functions; zeros of linear and quadratic functions.)

19. Arithmetic of Inequalities

($x < y$; order relation properties reviewed; $x < y, a < b \rightarrow x + a < y + b$, etc.; absolute value; the algebra of intervals.)

20. Statistics

(Average values and means; frequency diagrams; small and large populations; continuous distributions.)

21. Algebraic Structures

The idea of an abstract algebraic structure is introduced through many examples, some old, some new.

Single operation

Tables of arbitrary construction, truth table, modular arithmetic (one operation at a time); etc.

Commutative and associative properties, with review of old results.

Geometrical transformations and symmetries of simple figures; rotation, reflection, etc.

Two operations

Illustrations of combination of two simple structures with introduction of the distributive property.

Modular arithmetic; fixed points of $x \rightarrow ax + b$ in $\mathbf{Z}_n$.

Small finite fields.

22. Geometry and Measurement of Area and Volume

22a. *Geometry.* (Nonlinear properties; mensuration formulas for rectangles, triangles, parallelograms, circles, spheres; area and volume under similarity; trigonometric ratios; trigonometric functions and applications.)

22b. *Measurement of area and volume.* (Standard units; estimation; error; approximation.)

23. Number Theory

(Factors; primes; g.c.d. and l.c.m.; simple linear diophantine equations; Pythagorean triples.)

The list of topics above is certainly not complete. There is great flexibility in the amount of number theory which can be introduced effectively at this stage. Time and the interest of the teacher will be major determining factors, but the topics above are the ones which have close ties with the pupil's experience and intuition.

24. Geometry

(Pythagorean theorem as an area problem; ruler and compass constructions; conic sections; loci and envelopes and their constructions, simple properties, connection with celestial orbits; intuitive introduction to vectors, multiplication by scalar, and addition, based heavily on earlier experience.)

25. Irrationals and the Real Number System

(The triple (1, 1, $\sqrt{2}$) as a Pythagorean triple; irrationality of $\sqrt{2}$; successive decimal approximation; infinite decimals; $\sqrt{2}$, $\sqrt{11}$, $\sqrt{3}$, etc.; the full number line; the real numbers as an algebraic system — looking back at topic 15a.)

Essay 2 Teaching Mathematical Reasoning

1. Introduction

In the modern concept of mathematical education, the student learns by discovery. First isolated facts appear; these are seen to be related, and finally, general properties become apparent. How to help the student comprehend the general results is a very difficult teaching problem; the earlier procedure of teaching manipulation through quantities of drill involved no such difficulties.

Some ways in which present-day texts fail to clarify matters are given in Section 2. That the elements of logical reasoning in fact come easily to the student should be apparent through the illustrations in the next section. Some examples showing how general logical concepts may be brought out through simple examples will be found in the final section.

The essential thing is that the student not get wholly lost at some point, give up the attempt at real understanding, and simply try to learn to say things in the way the teacher likes. This requires patience in studying simple examples, and going back to them at later times to see better how they relate to other examples. The student must be kept aware of the difference between informal discussion and exact reasoning.

We shall talk mostly in terms of the written word, since a text represents authority to both teacher and student. A given text may, on perusal, seem clear and helpful; yet on a close examination, it is apt to be often meaningless and confusing (in spite of its having been "tested through classroom use"). Some texts use the "deductive method," with strict reasoning, typical of a graduate course; the student might understand the details, but he will miss the intuition and inspiration needed to make mathematics real and useful.

2. The Need for Reform

Sometimes individual statements are in fact meaningless, and sometimes what is said in one place simply does not jibe with what is said in another.

a. Texts often have a chapter on logic. In particular, one learns the logic of propositions. Applying this to another part of the book, we examine the statement

$$\text{If } x \neq 0 \text{ then } x/x = 1. \qquad (1)$$

This is of the form "if P then Q." Suppose P is false, i.e. x is 0. Then what is x/x? Q is meaningless, unless P is true. Applying the truth tables somehow does not work. Let us try something else: Form the "contrapositive." This gives:

$$\text{If } x/x \neq 1 \text{ then } x = 0. \tag{2}$$

The student is hopelessly lost. He begins to think that mathematics is in some manner a fraud. (The poorer student will not run into this problem.)

b. A standard statement in the discussion of sets is: "Two sets are equal provided that each element of one set is also an element of the other." What can this mean to the student? He first pictures two sets, and then tries to make them "equal." But if they are in fact the same, there is no reason to talk about them as though they were different, and no test makes sense. Why take an element of a set and see if it is in the set?

The basic trouble is that one is talking theoretically, out of any context; the student is not discovering though actual examples. Here is a possible example. Let A be the set of positive even numbers less than 10, and let B be the set of numbers which differ from 5 by 1 or by 3; then $A = B$. We have two different *descriptions* of the same set.

c. Addition is introduced as a binary operation. Later, it is said that subtraction is the inverse of addition. How can a binary operation have an inverse? In this connection, see the discussion of functions below.

d. The term "equation" is sure to appear in many places, with various meanings. The student may not get too confused, *unless* it is stated somewhere exactly what an "equation" is. The trouble is that the text does not differentiate between informal discussion and exact statements. Using "equation" to help give him an idea of the subject matter does not hurt, if he knows that this is the case. Looking for exact meanings where they are not as stated is obviously harmful. The same discussion applies to "open sentence" for instance.

e. Especially in older texts one may find statements like the following:

$$3x - 5 = x + 3: \qquad \text{an equation in } x. \tag{3}$$

$$y = x^2 - 2x + 15: \qquad \text{gives } y \text{ in terms of } x. \tag{4}$$

$$x + y = y + x: \qquad \text{an identity.} \tag{5}$$

$$\frac{x}{a} + \frac{y}{b} = \frac{x + y}{a + b}: \qquad \text{wrong.} \tag{6}$$

The reader is expected to see an expression and to deduce what the subject matter is and what kind of role the symbols play, etc. Why is it improper to tell him?

f. Texts commonly speak of rules, laws, facts, postulates, etc. without saying what they mean by these terms. How can the student understand what is meant?

g. Certain words are apt to be defined, like "or" and "if . . . then . . . ."

On the other hand, others, like "but," are left to him. Are all such words just used informally?

h. Most texts which define groups give statements of this nature:

$$\text{There is an element } e \text{ such that } ex = xe \text{ for all } x. \tag{7}$$

$$\text{For each } x \text{ there is a } y \text{ such that } xy = yx = e. \tag{8}$$

This implies that "existence" in (7) means "unique existence," thus weakening the student's concept of this quantifier. It should be specifically stated that a certain definite element e is assumed to have both properties (7) and (8).

3. Logic in Elementary Mathematics

One often speaks of "logic" as though it were something extra, outside of mathematics, which surrounds mathematics and bakes it into a whole; and that ordinary reasoning is only accurate if replaced by formal manipulation based on postulates. If one is studying a simple structure and wishes to use formal logic, then the postulates for formal logic must first be proved!

We shall use "logic" in its general sense: communication in precise terms. In examining simple examples, reasoning about structures is seen to be just expressing what is clear about them. Here are some illustrations easily comprehended by the student.

i. *Statements: truth and falsity.* "$18 + 7 = 24$." "$18 + 7 = 25$." The first statement (or expression) is labeled "false"; the second, "true." Should one search for a deep philosophical meaning of truth and falsity to be sure of this? No; we are in a simple situation where the attaching of one of two labels is part of the structure. One could use the labels "no" and "yes," or "bad" and "good."

j. *Deducing facts.* Daddy comes home, and says to Johnny, "Which hand?" Johnny tries the left, and finds nothing. He concludes that there is something in Daddy's right hand. His reasoning is:

right or left; not left; hence right.

No formal logic is needed to obtain the conclusion.

k. *Universal quantifiers.* "$a + b = b + a$, for all a and b." This first means something to the child as applying to a dozen or more numbers. Gradually the idea of "universality" appears. This may be viewed in two ways (both of which could well be stressed): There is a certain (infinite) set of pairs of numbers; the property holds for all at once. Or, at any time a pair of numbers is chosen, the property holds for these.

l. *Implication.* The student thinks the following may be true:

If a number is divisible by both 2 and 3, it is divisible by 6. (9)

To test this, he may try 2, 4, 6, 8, 10, 12, 14. Only 6, 12, . . . are divisible by both 2 and 3, and these are divisible by 6. His interpretation of (9) may be phrased as follows:

In the set of numbers divisible by 2 and 3, all are divisible by 6. (9′)

Now let us drop the quantifier. "If Beth's number is divisible by both 2 and 3, it is divisible by 6." What does this mean if Beth's number is 8? The statement does not apply; it need not have been said. If one must assign it a truth value, it is certainly not false; hence it is true. But for the present purposes, a truth value need not be assigned.

Note that statement (9) has a perfectly good meaning to the student: He looks at all nonzero numbers, and notes that each divided by itself is 1.

m. *Existential quantifiers.* Is 15 prime? Trying this, one finds 3, also 5; thus 15 is not prime. How about 17? Trying 2, 3, 4, . . . , 16, all fail; 17 is prime. The practical meaning of "existence" is perfectly clear. Is 42 divisible by 6? One discovers 7: $6 \times 7 = 42$. Unique existence has also a definite meaning. Note the different possible phrasing without a quantifier: $42/6$ is a natural number.

n. *Ordered pairs.* If one number is greater than another, you can subtract the second from the first. Note that "ordered" means simply that you know which is which; no relative position on paper is implied.

o. *Proofs by contradiction.* If n^2 is odd then n is odd. For take any n such that n^2 is odd. If n is even, we may write $n = 2m$; now $n^2 = 4m^2$, which is even, a contradiction. The analysis of this proof is as follows: If A is the set of n for which n^2 is odd, and B is the subset for which n is odd, then $A - B$ is void; hence $B = A$. The term "contradiction" has no deep significance; it is part of the simple language used in carrying out the proof.

p. *Constants, variables.* The student easily thinks of a letter as denoting a number. How about a "variable"? "A variable is a letter denoting an element of some domain." If he is told this, is it something he will use in proofs? Take (o) for instance. He may think of the proof as follows: Suppose someone had given him a number, and called it n; he is told that n^2 is odd. He shows that n is odd. Since he can always do this, the fact is true. Note that "n" is now a constant in the proof. For further discussion of this point see (r) below.

4. Teaching Concepts and Reasoning

In studying a simple problem, what is happening may be intuitively quite clear. Further examination should in no way replace the intuition; it should serve to clarify and strengthen it. One must be able to talk freely about a problem, in an informal manner, without committing oneself too specifically. When precise statements are made, their mathematical content should be clear. In terms of mathematical content, "if n is even then n^2 is even" and "if n^2 is not even then n is not even" say the same thing, and need not be differentiated; on the informal level, however, in discussing matters they may play different roles. Keeping the level of the discussion always clear to the student is most important.

We show here some ways in which the introduction and discussion of concepts can aid the student's understanding.

q. *Functions.* Admitting the great importance of this notion, one commonly recommends early introduction of the idea, with plenty of varied examples. The purpose seems to be to overwhelm the student. Better would be to come to the notion when it has some usefulness. At the start, the term may be used informally to suggest the idea of a correspondence. One place where one wishes a term to name the idea is in the discussion of subtraction; see (c). Subtracting 5 from a number undoes what adding 5 does. To understand this further, $f(n) = n + 5$ is introduced; this is a function (for example, over the natural numbers) whose inverse is the function under consideration. For another example, linear functions have some important elementary properties; these distinguish them from other functions. This statement requires the term "function" for its expression.

r. *Solution of equations.* We first take a concrete case, logically simple. Tom took a number; he squared it and added his original number; he got 6. Can we tell what number he started with? We can quickly try numbers, and find 2. Further searching may turn up -3. Now let us apply algebraic methods. Let us denote the number by x; we are given that

$$x^2 + x = 6. \tag{10}$$

Since $x^2 + x$ is 6, subtracting 6 gives 0:

$$x^2 + x - 6 = 0. \tag{11}$$

Using two distributive laws, we verify that

$$(x + 3)(x - 2) = 0. \tag{12}$$

The product of two nonzero numbers is not zero; hence

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0, \tag{13}$$

and

$$x = -3 \quad \text{or} \quad x = 2. \tag{14}$$

Thus there are no further possibilities.

Suppose Tom said his number was positive. Then $x \neq -3$, and this, with (14), gives $x = 2$. The reasoning is the same as in (j).

Now consider another problem: Solve the equation (10). This needs clarification. For instance, consider the domain of real numbers; we wish the solution set of (10) in this domain. The steps from (10) to (14) are now logically more complex. Each is an equation with a solution set; we must relate these. We cannot get from (10) to (11) by subtracting 6 from a number, since $x^2 + x$ is not now a number. (One should note that in the present interpretation, a different "variable" may be used in each equation.)

The simplest approach to the new problem is as follows: Assume that x is a number for which (10) is true. (Now "x" is a constant instead of a variable.) Then we derive (14), as before, and conversely. Thus (10) and (14) have the same solution set.

s. *Information, hypotheses.* The most useful way to think of a statement is as a means of imparting information. Thus "$a = a$" tells us nothing; "$a < 10$" tells something; "$10 \leq a \leq 10$" tells us a lot, in fact the same as "$a = 10$"; and "$10 < a < 10$" tells us too much: it is impossible.

When Tom said his number was positive, we thought:

$$\text{If } x > 0 \quad \text{then} \quad x = 2. \tag{15}$$

This seems peculiar: From the weak information $x > 0$ we derive the strong information $x = 2$. How can this be? We already have other information; with this, $x > 0$ and $x = 2$ are equivalent. Thus (15) cannot be considered as a statement by itself; it has meaning and is true because it is understood along with previous material.

This point is illustrated by (1), as follows. The part $x/x = 1$ has meaning, because of the hypothesis $x \neq 0$ just before it. (In an actual exposition, one must know whether "x" retains its meaning and the hypothesis "$x \neq 0$" is continued after the statement is made.)

Here is another obvious kind of example. The teacher writes

$$a \cap b \subset a \cup b \supset a. \tag{16}$$

Some students will be quite puzzled. One will say, "a and b aren't numbers, they are sets." If this is so, then with the usual meaning of the symbols (16) is a true statement. Otherwise, it may be meaningless. (When we considered the "statement" $a < 10$ above, we failed to state that a is a number.)

t. *Studying non-existent things.*

THEOREM. $\sqrt{2}$ is irrational.

Proof: For otherwise, we could write it as a/b, with a smallest a. Then $a^2 = 2b^2$; hence a is even, and for some c, $a = 2c$; now $b^2 = 2c^2$, contrary to the minimal property of a.

The student can certainly follow the details of this proof. But the whole may be quite mystifying. In particular, how do we actually find c, when a really did not exist? How do we square numbers, etc., which do not exist? For better understanding, one might try a wholly different approach.

Let us hunt for integral solutions of $a^2 = 2b^2$. Write down a list of squares:

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad \ldots$$

The pair 25, 49 is a near miss. We must look further. A slide rule will help us. Put 1 on the B scale under 2 on the A scale, and look for coincidences on the C and D scales. The following will be noted:

$$29, 41; \qquad 41, 58; \qquad 70, 99.$$

(Also 14, 19.8; 87, 123; 99, 140 could be discussed.) Squaring each and multiplying the first of each pair by 2 gives

$$1682, 1681; \qquad 3362, 3364; \qquad 9800, 9801.$$

Does this give us any ideas? Someone may note that the first number is always even; hence the second one must be. Another curious happening is the double occurrence of 41. Why? If the second pair had worked, so would the first. Working with these ideas, a proof like the former proof soon appears. Moreover, we see that what has really been proved is: *Theorem.* Any rational number, squared, is not 2. In this version, there is no question of working with non-existent numbers. Is not the first proof then good? Sure; with the *assumption* that $\sqrt{2}$ is rational, we had numbers a, b, and c with certain properties. Again, statements have meaning in the light of what is assumed.

u. *The deductive method.* Proving facts about numbers through the use of the CAD laws has a fatal drawback: the facts are obvious anyway, often as much as the CAD laws themselves; hence the meaning of "proof" is lost. Moreover, this can hardly illustrate the deductive method, since the system of numbers is thought of as known to the student, and hence results are not consequences of given hypotheses. The usual logical reasoning is illustrated by theorems from number theory for instance. For a whole "method" one must look elsewhere. One could start with the axioms for a commutative group, and assume there is a single generator; there is real interest in finding what the possibilities are.

v. *Some subtle logical questions.* This is a word of warning that, if one requires a strict logical development throughout, some curious questions will arise. For instance, suppose we know that

$$\text{For all } a \text{ and } b, \text{ if } ab = 0 \text{ and } a \neq 0, \text{ then } b = 0. \tag{17}$$

We wish to show that

$$\text{For all } x, \text{ if } x^2 = 0, \text{ then } x = 0. \tag{18}$$

Take any x. Using $a = b = x$ in (17) gives

$$\text{If } x^2 = 0 \text{ and } x \neq 0, \text{ then } x = 0. \tag{19}$$

If only we also knew that x is not 0, we could prove that x is 0! To understand this fully requires real confidence in the logical structure.

Consider any proof by contradiction. We have a system with certain hypotheses, say $H_1, H_2, \ldots, H_n$; we wish to prove K. Assume not-K. With the enlarged list we prove not-H_1, a contradiction; we say: hence K holds. Note that with the enlarged hypotheses, H_1 (in fact, every statement) is both true and false. What happens to the truth tables in this case? The answer is: any way we can use truth tables, etc., with the knowledge at our disposal to help us arrive at a contradiction is all right.

Essay 3 Some Thoughts on Algorithms for Elementary Mathematics Specialists

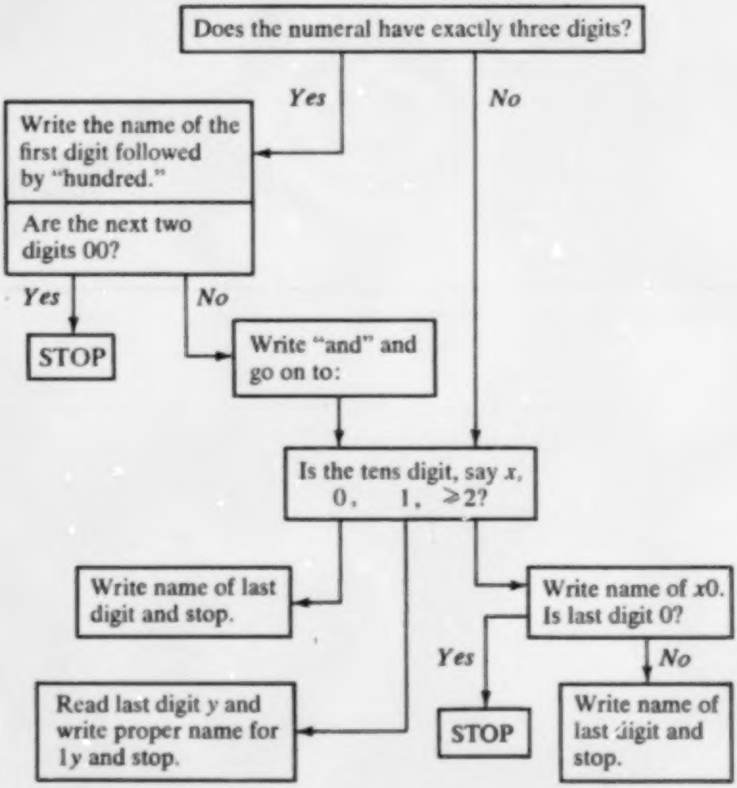
Since we are not yet thinking seriously of teaching computing (computer methods) in the elementary schools, we should perhaps first explain why algorithms for the elementary mathematics specialist are a significant problem. Well, we are thinking of teaching some computing in the secondary school, and we might as well lay the proper groundwork — build up some experience and feeling for the ideas — as long as we have the opportunity. Next, we do teach a number of algorithms in the elementary school, and maybe it would be helpful to be aware of them *as* algorithms. Finally, the teacher and the mathematics specialist may plan a presentation in a much better way, be much more aware of the difficulties that the student may face, and see the pitfalls and special cases more clearly, if he has thought about the ideas to be presented in an algorithmic way. This preparation is quite akin to the completeness and clarity of exposition which come from planning materials for programmed instruction under careful computer control, whether the ultimate product is to be prepared in book or in computer form.

A few examples of relevant algorithms should be mentioned. Early in the primary grades children have to learn to read numerals. There are often temporary difficulties with this — confusions between 13 and 31, between 605 and 65, etc. It is a real help in foreseeing such difficulties, and in organizing the coverage of the problem, to draw a flow diagram. The basic difficulty, of course, is that the English language does not fit a place-value system, whereas written numerals do. Let us take the simple case of numerals between 1 and 999, and hope that we can draw a correct flow diagram.

What we assume is the ability of the machine to print, on command, the basic distinct building blocks we need; to wit, the English names for the numerals 1,2,3, . . . ,19, for 20,30,40, . . . ,90, and the words "hundred" and "and." A diagram appears on the next page.

You see that this diagram is not trivial, and that the places of branching are points of pedagogic importance and potential student difficulty. It is fun to extend this further, even to decimals, and finally to realize in just what way the English expressions for .101 and for 100.001 differ.

Other algorithms close to the elementary school are those for long multiplication and long division. There is an interesting problem, much studied in the 1920's, in the alternative algorithms for subtraction, and in seeing how


Does the numeral have exactly three digits?
Yes
No
Write the name of the first digit followed by "hundred."
Are the next two digits 00?
Yes
No
STOP
Write "and" and go on to:
Is the tens digit, say x, 0, 1, ≥2?
Write name of last digit and stop.
Write name of x0. Is last digit 0?
Yes
No
Read last digit y and write proper name for 1y and stop.
STOP
Write name of last digit and stop.

these affect the possibilities for division. The process of factoring a positive integer into prime factors, divisibility tests, and the building of new such tests out of old — say for divisibility by 12 out of those for 3 and 4 — are good algorithmic material.

Removal of parentheses in an arithmetic or algebraic expression and the more general problem of simplification can be looked at algorithmically, although the full pattern is really very difficult.

There are, of course, many more algorithms more appropriate to the secondary school, and even to high-school computing courses, but they may still be valuable to the elementary specialist. Examples are a real understanding of the algorithm for writing down the solutions for a quadratic equation. Consider $ax^2 + bx + c = 0$. A recent paper by George Forsythe gives a wealth of illuminating detail on this problem. Here are a few aspects of it. Some of the branches: (a) If $a = 0$, not quadratic. A computer doesn't know that without being told. (b) a is *almost* 0, relative to b and c. This assumes the problem has been scaled somehow. Then the quadratic formula loses all accuracy. Instead, the first guesses might be $-c/b$ and $-b/a$. What will the roundoff errors do? (c) The discriminant is 0 or, worse, almost 0, in which case you have to decide what to believe.

Then there are examples of the nonnumerical applications of computing, which are going to become ever more important. How do you sort numbers, anyway? Sort and then point-count a bridge hand? Find the convex hull of a given set of points in 2- or 3-space? Find the triangles bounded by a given collection of lines?

Essay 4

Applied Mathematics for Elementary Mathematics Specialists

The mathematics specialist for kindergarten through the sixth grade should be familiar with several areas of applied mathematics. Mathematical applications furnished by the life and social sciences are taking exciting new directions, and several examples will be cited that can be adapted for classroom use in the sixth grade. It is of paramount importance that the elementary school pupil be introduced to the computer in some form and be led to see it as a revolutionary tool of future mathematics. Classroom discussion, even in Grades 4-6, should attempt to indicate the relevance of the computer to all kinds of mathematics and should attempt to formulate some ideas and concepts involving standard number systems. There is no need to emphasize the classical areas of engineering and physical science as sources of mathematics problems; moreover, adapting problems from those fields for the elementary classroom is difficult.

Whereas most mathematical developments of the 19th century were inspired by the study of deterministic phenomena in the physical sciences, it is manifest that a great deal of evolving new mathematics is motivated by questions coming from the social, managerial, and biological sciences. A key ingredient of these disciplines is the intrinsic element of uncertainty and variability. In order to deal quantitatively with such problems it is crucial to learn the intuitive nature of probability theory, as well as to understand its precise construction. We strongly recommend probability and the elementary theory of stochastic processes as a fundamental core course for a mathematics specialist. The concept of randomness is difficult for people to absorb because of their previous exposure to the deterministic aspects of physical science, but it can be mastered by appropriate study and practice with various models in biology and social sciences. It involves a language and process of thinking all its own and somewhat different from the standard mathematical methodology.

Some knowledge of statistical procedure is desirable also in handling data for testing the adequacy of model-building, a well established technique in applying mathematics in all the sciences.

Examples of the kinds of mathematical applications in the biological sciences that possibly could be adapted for Grades 5 and 6 follow. It should be emphasized that only a minimal background in biology is needed.

Population Biology

The Hardy-Weinberg Law. (Hardy was a famous English mathematician, Weinberg a German geneticist.)

Although the chromosomes inside a cell are complicated chemically, there are some characteristics (blood type, susceptibility to specific diseases, color in sweet peas, height of corn stalks) which can be shown to be determined by small areas on the chromosome pairs. We assume that the biological characteristic under investigation is governed by two alternative types of genes (called alleles), each of which occurs at a specific place on a different member of the chromosome pair. We label the two alternative possibilities A and a. Then there are three possible pairs AA, Aa, and aa. These are called the genotypes and, in general, are associated with different characteristics (e.g., different colors in sweet peas).

We now consider only these pairs, and in particular what occurs when one generation reproduces to form another. This reproduction occurs through a crossing procedure resulting in the following pairings: AA and AA, AA and Aa, AA and aa, Aa and Aa, Aa and aa, aa and aa, which may be looked on as "matings." Our model postulates that the offspring forming the new generation are formed by choosing one of the genes at random from each of the parental genotypes in the mating. Hence if we look at the $Aa \times Aa$ mating, to form an AA individual we must choose A from the first (probability $\frac{1}{2}$) and A from the second (probability $\frac{1}{2}$); the probability that AA is formed is therefore $\frac{1}{4}$. For the Aa offspring, we could choose A from the first and a from the second or the other way around. Each has probability $\frac{1}{4}$ so Aa has probability $\frac{1}{2}$. In the same way as the AA, it is easy to see that the chance that aa is formed is $\frac{1}{4}$.

The teacher could work out with the class other similar calculations, which come under the heading of segregation.

Returning to the three genotypes AA, Aa and aa, we suppose that they occur with relative frequencies u, v, and w respectively with $u + v + w = 1$ and $u \geq 0$, $v \geq 0$, $w \geq 0$. We postulate that the mating is at random; in the case of a two-sex population this merely means that any one individual of one sex is equally likely to mate with any one individual of the opposite sex. Thus the frequency of any one of the above six matings is determined only by the frequency with which the genotypes occur. So, in our present case, the frequency of the $AA \times AA$ mating, for example, is $u \cdot u = u^2$. (Of course in a two-sex model this assumes that the genotypic proportions in both sexes are the same.) A table of random mating frequencies appears at the top of the next page.

Now we require the frequencies of the three genotypes in the next generation. First, how may the AA offspring be formed? All the offspring of the $AA \times AA$ mating will be AA — a contribution of u^2 from Table 1. Also, the frequency of the $AA \times Aa$ mating is $2uv$, but only one half of the offspring are AA — i.e., a proportion uv. Finally the $Aa \times Aa$ matings occur

Table 1

Female (or second individual)	Male (or first individual)		
	AA u	Aa v	aa w
AA u	u^2	uv	uw
Aa v	uv	v^2	vw
aa w	uw	vw	w^2

with frequency v^2 and since one quarter of the offspring are AA we have a contribution of $v^2/4$. The total frequency of AA produced is then $u^2 + uv + \frac{1}{4}v^2 = (u + \frac{1}{2}v)^2$ since these are the only matings which can produce AA.

It would be very instructive for the teacher to justify all the entries in the following table.

Table 2

Type of Mating	Frequency of Mating		Offspring	
		AA	Aa	aa
$AA \times AA$	u^2	u^2	—	—
$AA \times Aa$	$2uv$	uv	uv	—
$Aa \times Aa$	v^2	$v^2/4$	$v^2/2$	$v^2/4$
$AA \times aa$	$2uw$	—	$2uw$	—
$Aa \times aa$	$2vw$	—	wv	wv
$aa \times aa$	w^2	—	—	w^2
Totals	1	$(u + \frac{1}{2}v)^2$	$2(u + \frac{1}{2}v)(w + \frac{1}{2}v)$	$(w + \frac{1}{2}v)^2$

Thus the frequencies u_1, v_1, w_1 of the AA, Aa, aa in the next generation are respectively

$$(u + \tfrac{1}{2}v)^2, \qquad 2(u + \tfrac{1}{2}v)(w + \tfrac{1}{2}v), \qquad (w + \tfrac{1}{2}v)^2.$$

If we repeat this procedure for another generation we find $u_2 = u_1$, $v_2 = v_1$, $w_2 = w_1$; that is, a stable situation is achieved. This result is known as the Hardy-Weinberg Law. The analysis could first be done numerically and by simulation, using a table of random numbers.

Abundant illustrations of the significance of the Hardy-Weinberg Law in medical counseling, evolutionary studies, and so forth are obtainable. It should perhaps be pointed out that although the Hardy-Weinberg Law is such a mathematically appealing phenomenon one can contrive many other models which do not obey the law at all.

A little statistical estimation can be introduced at this point. Perhaps it would be worthwhile to illustrate with Mendel's original experiments. The original paper of Mendel is a model of scientific thinking and should be read also.

Mutation and Growth

For this example we can consider a population consisting of two types of individuals A and B, which, as we begin the investigation, are in proportion p_o, $1 - p_o$. We let each type produce only its own kind. To introduce mutation, we postulate that each individual of type A has probability μ of changing into a B in the next generation while B mutates to A with probability ν. If we look at the offspring of the nth generation when the frequencies of the two types are p_n, $1 - p_n$ we find that

$$p_{n+1} = (1 - \mu)p_n + \nu(1 - p_n)$$

since $(1 - \mu)p_n$ will be the proportion of A genes not mutating to B and $\nu q_n = \nu(1 - p_n)$ is the proportion of B's mutating to A. This serves as an introduction to a simple dynamic linear recursion relation. It is obvious that when $p_n = \nu/(\mu + \nu)$ there is no change from one generation to another and an equilibrium is attained. This is another type of phenomenon occurring in other parts of applied mathematics.

Migration and Growth

For this model we assume that there are two populations. One has two types A, B with frequencies p_n, $1 - p_n$ in the nth generation. The other has constant A-gene frequency P and B-gene frequency Q. We postulate that a proportion m of the first population is replaced in each generation by individuals from the second. We then have

$$p_{n+1} = (1 - m)p_n + mP,$$

another simple recursion relation. One can easily combine the mutation and migration models. Of course, both models involve geometric convergence of series for a complete solution; this may be beyond the sixth grade pupil but an intuitive description of the limiting behavior is probably warranted.

Selection and Growth

The basic ideas of Darwin's theory of natural selection are almost universally known. We talk about some individuals being "fitter" than others; i.e., they have genes which give them certain features enabling them to be superior in some way or other to the rest of the population. For a mathematical treatment, "selection," as this process of choosing the fittest is called, means that either certain individuals produce more offspring or that the offspring of these individuals have a better chance of surviving to reproduce and contribute to subsequent generations. The example we give is of "gametic" selection which reflects the relative advantage of some gametes over others. (Gametes can be taken to be the 'A' or 'B' singletons from which a new generation is formed.) This model introduces nonlinearity and is mathematically more interesting than the one above.

Instead of treating the parents as genotypes, we regard the parental generation as a pool of gametes (A and B) which form new gametes, which in turn pair off to become the next generation. Here gametic selection involves one type (say A) producing a higher proportion of descendents in the next generation. Suppose the proportions of A and B production are $1 : 1 - s$ $(1 > s > 0)$. If the frequencies in the nth generation of A, B are p_n, q_n respectively, then it is not difficult to see that

$$p_{n+1} = \frac{p_n}{p_n + (1 - s)q_n} = \frac{p_n}{1 - sq_n}$$

$$q_{n+1} = \frac{(1 - s)q_n}{p_n + q_n - sq_n} = \frac{(1 - s)q_n}{1 - sq_n}.$$

These relations are already more complicated than the earlier ones. The complete analysis is once again not difficult if we consider

$$\frac{q_{n+1}}{p_{n+1}} = (1 - s)\frac{q_n}{p_n} = (1 - s)^{n+1} \cdot \frac{q_0}{p_0}.$$

And since $(1 - s) < 1$, the ratio of q_{n+1} to p_{n+1} is decreasing to zero and we see that q_{n+1} must be likewise decreasing to zero; in other words the 'B' gamete is becoming extinct. This is a very intuitive result and has the advantage that examples are easily found of such selective changes in various animal populations.

Other simple models of special breeding schemes can be introduced, for example, self fertilization. Many of these have very simple analyses along the lines of the previous models. As an exercise the teacher could work the analog of the Hardy-Weinberg Law for three alternative genes (or the six resulting genotypes).

There are several other applications of mathematics to genetics, many of which are quite recent. When we consider polyploid individuals, that is those having more than two chromosomes aligned in the cell nucleus, more difficult problems arise. We can also construct models where certain matings are not allowed to occur (incompatibility systems which have interesting analyses).

Other models involve simple applications of randomness and algebraic analysis, especially the study of nonlinear recurrence relations of prey-predator models with competition of species or epidemiology.

Coding Problem

Consider four distinct nucleotides for which each triplet determines one of twenty amino acids, which in turn build protein blocks. Several simple algebraic systems can be introduced here.

Some other areas of applied mathematics with which the teacher should have some familiarity are, briefly:

Managerial Sciences

1. Mathematical programming
   - a. Elementary versions of the traveling salesman problem.
   - b. Transportation problems.
2. Game theory*
3. An introduction to decision theory**

Physical Sciences***

Biological Sciences***

Computer Science

We strongly urge that the mathematics-specialist teacher acquire experience with the programming techniques of computers and acquaintance with their potentialities. Perhaps a course is desirable. It should include:

a. Introduction to one or two of the principal computer languages.
b. Simulation and Monte Carlo technique and their applications in science.
c. Elements of numerical analysis and algorithmic processes.

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* John Davis Williams, *The Compleat Strategyst*, McGraw Hill, New York, 1954.

** J. L. Hodges, Jr. and E. L. Lehman, *Basic Concepts of Probability and Statistics*, Holden-Day, San Francisco, 1964. H. Chernoff and L. Moses, *Elementary Decision Theory*, John Wiley, New York, 1959.

***No suitable book is available, unfortunately.

Essay 5 | Experiment in a Model City

A variety of projects will be required to carry out the recommendations of this report. There will have to be subsidized projects for the writing of texts and other materials and for experimental teaching. Regional conferences will have to be organized to bring mathematicians together with the various teachers and administrators who will be concerned directly in establishing the new curricula in the classrooms for prospective teachers. In-service and summer training institutes for teachers will be needed. All these projects will require financial support, but they are all for purposes which in the past have drawn generous support from both public and private sources. There is every reason to hope that the support will be continued.

But there was also a quite different proposal presented to the Conference, which considered the fairly desperate plight of the elementary teaching apparatus in the United States and looked for a means to get the country started toward a real solution on a national scale. The problems are of a magnitude calling for staggering investment in the school system.

To reduce the elementary school teacher's excessive teaching load of five hours a day to a reasonable three hours would multiply the total cost of salaries by $\frac{5}{3}$. This relief will give the teacher time to plan her lessons more effectively, to attend in-service institutes, and to study independently to improve her subject competence, etc. To cut the average class size from 30 to 20 pupils would increase the costs (including classroom construction) by $\frac{3}{2}$. To institute a modest professional salary scale — say, $6,000 to $15,000 a year at present — would surely require more than a $\frac{6}{5}$ increase over the present cost. Combine the increases ($\frac{5}{3} \times \frac{3}{2} \times \frac{6}{5}$) and you get three times the present expenditures for salaries and building costs alone. Could the American taxpayer be persuaded to invest in education on this scale? There is no doubt that the United States has the wealth for such a transformation but what about the will? Could a sufficiently dramatic example be devised to demonstrate in practice the enormous difference that adequate funds would make in our schools?

Suppose that a city of about 100,000 population were selected for the demonstration. It should be one whose school system was recognized generally as good by present standards but one beset with all the problems that have become characteristic of cities across the nation. Then suppose that outside funds could be obtained for a twelve-year period to be added to the regular school funds raised by taxation in the proportion of three to one!

An experiment on such a scale would turn the eyes of the nation to that city and bring education to the forefront of the national picture as never before. In twelve years one could hope that the pupils passing through this school system would make it evident, by their accomplishments and attitude, that the school budgets we now provide are woefully inadequate. Every parent in every state would demand for his own children the same educational advantages, and the requisite socio-political organization to channel the necessary fraction of the gross national product into education would quickly develop.

The success of such an experiment would call for wisdom to guide the expenditure of the donated funds, of course. The community of educators and scholars would respond with alacrity, but consummate skill in administration would also be needed. There are people who could rise to the challenge.

Some may contend that similar experiments already are being carried out in the school systems of rich suburbs, where the budgets may be twice the budgets of nearby urban areas, and that their schools and the intellectual accomplishments of the populace could be studied to evaluate the influence of schools on communities. This situation, however, is different from the one envisaged for the proposed experiment. The wealthy suburban school draws from a homogeneous student population, socially and financially secure and oriented toward education, a completely different population from that of the city. There could be no before-and-after here to be studied; through stability and wealth such communities have acquired gradually the qualities we seek for the model city. Those communities have produced the schools; in our proposed experiment, schools would produce the community.

Where is the money? The political impossibility of obtaining federal or state monies for the experiment is obvious, as is the economic impossibility of raising municipal funds. There remains the characteristic American answer — private funds. A cooperative combination of several of the large private foundations could manage such sums. The Cambridge Conference by itself is too specialized a group to broach the idea to the foundations, but it might take the lead in appealing to others whose interests, competence, and repute are such that they could properly put the questions.

Essay 6 What Is a Teacher?

Of course a teacher is someone who teaches — not just occasionally, but someone who works at teaching. And teaching is helping to learn. But what is learning?

Most people will agree that learning is an activity you have to accomplish by yourself. It is like eating. Someone can tell you where food is, or can set it before you — they can even put it in your mouth if you're a baby, say, or crippled. But the final act of eating you must perform yourself. And so with learning.

So helping someone to learn — that is, teaching — is a little like helping someone to eat. At the beginning a mother selects the food, buys it and brings it home, prepares it, puts it in the baby's mouth, wipes it off his chin, and puts it back in his mouth again. But her object is to get her child to eat independently. Ultimately he should be able to choose his own food to satisfy the requirements of both health and taste; he should be able to obtain, prepare, and eat his food himself.

And so, again, with learning. Unless a teacher helps her pupils to become independent of her, unless she consciously heads toward the day when they can do well by themselves, at choosing what to learn, at acquiring the necessary materials, and finally at learning, she will not succeed in the ultimate sense — even if her pupils have gathered much information while they are with her.

If we take this viewpoint seriously, it has far-reaching implications for the organization of our schools. It does *not* mean that the teacher just gives heavy homework assignments. Rather, it means that assignments are designed to lead to genuinely independent thought, and that the activities in the school itself are directed toward encouraging students to pursue individual interests, to make discoveries, to acquire a taste for study, to develop an ability to gather information, and to understand.

What gives to teaching its greatest challenge, and what makes its problems so vastly more complex than those involved in helping a child to eat, is the tremendous variation in the learning process from one individual to another. The scientific study of learning is barely beginning, but it is recognized generally that there is a variety of basic patterns of learning, and that superimposed upon these patterns are the individual ability levels distributed over what are probably thousands of separate characteristics which enter into the learning apparatus of a given personality. Over and above the differences in predisposition and capacity to learn, the learning process, we know, is highly sensitive to the total experience of an individual — his relations within the family and with friends, his contact with mass communication media, his reading, his dreams, his play. Our democratic aim of educating each person to the point where he can realize his fullest potential places upon the teacher a great responsibility; she should study and understand each one of her pupils as a distinctive individual, and devise ways of helping him to learn that which is peculiarly suited to his needs, his interest, and his abilities.

Side by side with the continuing effort to analyze her students and to understand them sympathetically, a teacher has the obligation to work continuously at the selection of the facts, techniques, ideas, and attitudes which she will ask her students to learn, and at the development of the methods and devices she will employ to help them accomplish this learning. In this respect we must understand clearly that the nature of our society and the role of the individual in it are undergoing certain revolutionary changes and are disrupting patterns which have been constant heretofore for generations, if not for centuries; and we must clearly see that these changes impose upon the teacher a concomitant pattern of new duties. Nowhere is this clearer perhaps than in the area of mathematical instruction — although in reality our developing ideas about physical and biological science, about the study of language (foreign and domestic), and the study of society and our developing attitudes concerning intergroup relations, impose demands on the teacher which are just as heavy though perhaps less clearly articulated.

What are the changes in mathematics itself which must be reflected in the elementary classroom? For one thing, the sheer volume of new mathematical research has increased year by year at a sharply accelerating rate, and a significant fraction of this work affects our understanding of the most fundamental concepts. In direction, mathematics has become much more abstract and, paradoxically, because of this abstraction it has become applicable to, and has derived sustenance from, a much wider range of applications. From the study of numbers and geometric figures it has broadened its scope to include every domain where form and structure can be discerned. Finally, the art of computation has become infinitely more complex, and the practice of it has shifted the routine burdens of execution to electromechanical devices while demanding much more in the way of control and design from the practitioner.

All these developments require not only that the teacher must alter at this time the mathematical curriculum which has heretofore remained static but also that she must continue to alter it from year to year throughout her teaching career. They mean, too, that the teacher of elementary mathematics must work not merely at training students to follow and apply prescribed computational routines, but also at getting them to understand abstract concepts to the point where they can devise and test new computational routines; she must stimulate them to formulate new concepts arising from diverse realms of experience and to search for the properties which relate these to concepts; and she must educate them to employ relatively sophisticated patterns of mathematical language so that they can communicate freely about their work.

These multifold obligations which we are delineating for the elementary school teacher entail two principal positions about the individuals to whom we assign this work. In the first place, the amount of specialized knowledge, ability, and interest required to carry out these tasks in even a competent, not to say inspiring, manner, obviously transcends what can be expected of a single individual. We must begin to think of the elementary teaching corps as composed of a variety of persons contributing in diverse ways to a common goal. And, in the second place, we must recognize that the satisfactory discharge of her duties requires of a teacher many kinds of professional activity other than direct contact with students, and we must provide such working conditions and environment as will facilitate the prosecution of these activities. Let us examine these two propositions separately.

The idea of specialization among elementary teachers at first suggests the kind of instructional pattern now found at the high school levels. Most educators consider this pattern unsatisfactory at the elementary levels. Actually, however, the concept of specialized teaching is compatible with a host of instructional patterns quite different in character. For example, insofar as we require each child to be the subject of careful and sympathetic observation designed to discover optimum methods for developing his individual potential, it seems clear that each child should be associated with a single teacher, say *A*, who can work with and study him in a variety of learning conditions and subject areas. At the same time, since *A* cannot be expected to be expert in all the everchanging curricular areas, we might wish a second teacher, *B*, to divide her time among a larger group of children, seeing each child for only a short part of the day for the purpose of dealing specifically with a single subject area, such as mathematics. And of course we would expect *A* and *B* to work together, with respect to their common students, by intercommunication and joint projects. There thus emerges an instructional pattern very different from what we find in high schools and colleges, one in which a given pupil encounters more than one teacher simultaneously at certain parts of the day.

One implication of such a program is the prospect of a teaching force sharply increased in size. Such an increase may, in fact, prove necessary or desirable, but it is by no means required by a scheme of cooperative teaching. Of course, if our view of the teaching environment is restricted to the conventional, closed classroom containing thirty seats facing the front of the room, then simple arithmetic will show indeed that to have more than one teacher in a room at one time requires an increase in the total number of teachers. On the other hand, reverting to our early desideratum of independent, individual learning, we can imagine a school in which single students, or small groups, work by themselves for significant periods of time in semiprivate areas within open rooms of considerable size with several teachers, including *A* and *B*, circulating and providing help and observation in a variety of ways. Such a program might well be feasible with no increase in total size of instructional staff.

It should be emphasized that there is wide latitude for experimentation in the cooperative teaching patterns which may be evolved to accommodate specialized teaching in the elementary school. The activities undertaken by elementary school teachers who have specialized in a subject area such as mathematics can range from dealing with individual students to conducting in-service courses for other teachers, to monitoring new text materials and innovations in technological teaching devices, to name but three examples; and we can envision groups of teachers each of whose activities are concentrated among a different few of the varied forms of specialization.

But let us look now at our second proposition: that an inventory of the new responsibilities being assigned to elementary school teachers requires us to provide adequate time and a suitable environment for a large variety of activities which do not involve direct contact with pupils. What does this mean in detail?

At present the normal distribution of an elementary school teacher's time involves seven hours a day at school: five are spent in direct supervision of students in class (or for brief periods on the playground), one is designated as a lunch hour (which often involves further contact with students in the cafeteria or even classroom), and one is spent in preparing the classroom before and after the students are in session and in a multitude of clerical tasks such as completing pupil attendance forms. Additional time in school is required on occasion to attend staff meetings, in-service training sessions, parent conferences, or PTA meetings. And work at home is expected to cover grading of student papers, preparation of lessons, and inspection of text books. Essentially no time is provided for teachers to discuss problems involving curriculum or methodology; no time, space, or materials are provided for individual study, and the desirability of visits by teachers to pupils' homes is ignored. It is fairly evident that the most dedicated and efficient of teachers can barely be expected to discharge even the most traditional instructional duties in a satisfactory manner within such painful circumstances. To expect her to measure up, under such conditions, to the kind of job we have outlined is absurd.

What sort of school environment can we imagine within which we could realistically expect a teacher to perform the tasks we have listed? Perhaps an average of three hours per day of contact with pupils is a reasonable goal at which to aim. Subprofessional teacher's aides must be engaged to provide clerical assistance and to free the teacher from routine chores so that she can concentrate on her primary work. Help with homework and playground supervision is desirable, though the teacher will wish to maintain some contact with these student activities. Space and facilities for individual study, or for conferences among small groups of teachers, must be available; the most basic library items, such as current journals in education and subject areas, should be on hand in a suitable room of the school; and time for access to more substantial library centers in the school district should be considered a normal part of the workday. Both subject specialists and general teachers will wish to study their individual students in depth, and each school indubitably will have ably manned facilities for psychological testing and research. Those aspects of instruction which can be made routine, such as drill in the application of algorithms, will be assigned to computer-based machines, so that the time of teachers may be conserved for the truly creative aspects of instruction. Films and projection rooms, as well as properly equipped laboratories for science, language study, and mathematics, will be among the teacher's tools. Administrative assistance and computerized aids will facilitate the testing and regrouping of students where instruction is organized in ungraded patterns. The teacher will be expected and encouraged to continue her studies in the material of her subject area as well as in methodology, and a regular system of sabbatical leaves will be available for those teachers wishing to deepen their penetration of some aspect of their work.

It will not have escaped the reader's attention that the creation of such a stimulating environment for teaching and learning is likely to involve considerable increase in the cost of education. To some extent the economic cost may be lightened because the assignment of routine instructional tasks to computer-based machines and the widespread and persistent encouragement of individualized self-study will tend to lower the teacher-pupil ratio. In this connection we may expect to see increasing use of students themselves as instructional aides, for learning-by-teaching is a phenomenon long familiar to teachers but not adequately exploited as a systematic method of instruction.

Any savings effected by reducing the teacher-student ratio will be more than offset, however, by the increased salaries which will have to be provided to assure an adequate supply of sufficiently talented individuals. The present miserable salary scales prevailing in most school districts take improper advantage of the fact that the great bulk of elementary teachers do not need to support a family with their income from teaching. But for the teaching profession of the future we shall have to attract many men and women who can fashion a full-time, lifelong career upon their work. Indeed, for this purpose we shall have to do much more than raise salaries. We shall have to create a structure within the profession of elementary teaching whereby merit is given due recognition, and wherein advancement from one level of specialized teaching to another, involving increased responsibility and rewards, can continue over a long span of years without being diverted from teaching to administration.

And so we return to the question with which we began — What is a teacher? In part the answer will be determined by the decisions of society, by the demands which it places upon teachers, and the support it provides.

In part the answer will be determined by the decisions of those individuals who dedicate themselves to a career of scholarship and of "helping to learn." It is not overly dramatic to say that these decisions will turn out to be critical in shaping the destiny of mankind.

It falls to us, the contemporary scholars and educators, to point out the possibilities to society and to inspire the individuals who will follow us.

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